ECE 201, Section 3
Lecture 21

Prof. Peter Bermel

October 12, 2012

Exam #2: Monday, October 15

- Runs from 6:30-7:30 pm in EE 129, unless specifically told otherwise
- Will cover Lectures 12-21:
  - Source transformation
  - Thèvenin and Norton equivalent circuits
  - Maximum power transfer
  - Inductors, parallel & series; LR circuits
  - Capacitors, parallel & series; RC circuits
  - Linear response; waveform generation
  - Undriven LC and RLC resonators

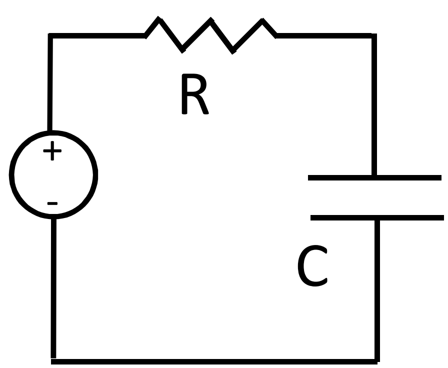
Exam #2: Monday, October 15

- Posted 4 sample exams on Blackboard – also includes equation "sheet" at bottom
- The one marked "Exam 2 – Review Session" from '98 will be reviewed tonight starting at 6:30 pm, here in EE 170

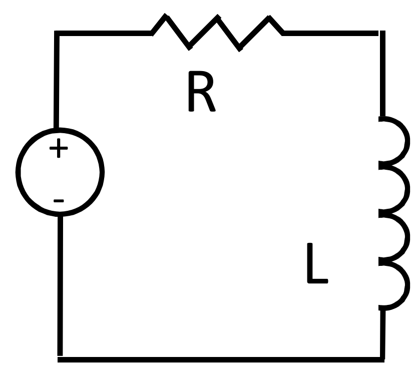
General Solutions: RC & RL Circuits

$$X = X_\infty + (X_o - X_\infty)e^{-(t-t_o)/\tau}$$

- Solution steps:
  - Choose X: RC circuits: X=Q or X=V; LR circuits: X=I
  - Find X_o (simplifying diagram)
  - Find X_∞ (simplifying diagram)
  - Find R_{th} (for inductor/capacitor)
  - Find τ: RC circuits: $\tau = R_{th}C$; LR circuits: $\tau = L/R_{th}$
  - If circuit changes at t_1, use $X(t_1)$ from prior solution for initial values



RC Circuit



RL Circuit

General Solutions: LC Circuits

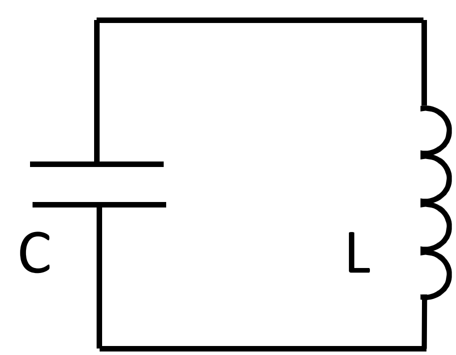
- For an undamped, free (undriven) LC circuit:

$$Q(t) = A\cos\omega t + B\sin\omega t$$

  - Q(t) is capacitor charge at time t
  - A and B determined by initial conditions
  - ω is the resonant frequency; $\omega = 1/\sqrt{LC}$
- Alternatively, can write:

$$Q(t) = A'\cos(\omega t + \phi)$$

  - ϕ is the phase angle



LC Circuit

Complex Exponentials

- Some key equations:

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

$$e^{(i\omega - \Gamma)t} = e^{-\Gamma t}(\cos \omega t + i \sin \omega t)$$

$$\cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

Trigonometric Identities

Can use Euler's equation to write:

$$
\begin{aligned}
e^{i(\alpha+\beta)} &= e^{i\alpha}e^{i\beta} \\
&= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&\quad + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)
\end{aligned}
$$

Which yields:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

General Solutions: LC Circuits

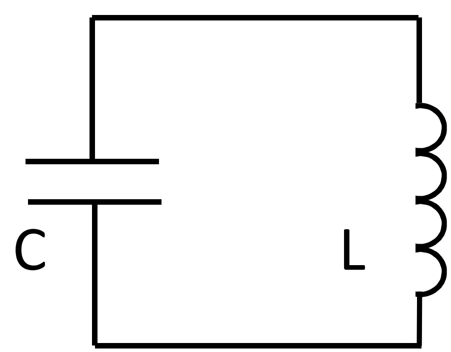
- Can rewrite our previous solution:

$$Q(t) = A\cos\omega t + B\sin\omega t = \mathrm{A}\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) + B\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)$$

$$= \left(\frac{A}{2} + \frac{B}{2i}\right)e^{i\omega t} + \left(\frac{A}{2} - \frac{B}{2i}\right)e^{-i\omega t}$$

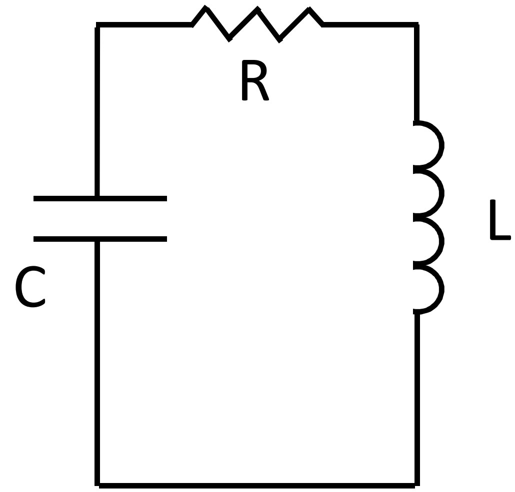
- We can define new coefficients in front to obtain:

$$Q(t) = A_+ e^{i\omega t} + A_- e^{-i\omega t}$$



LC Circuit

RLC Circuits



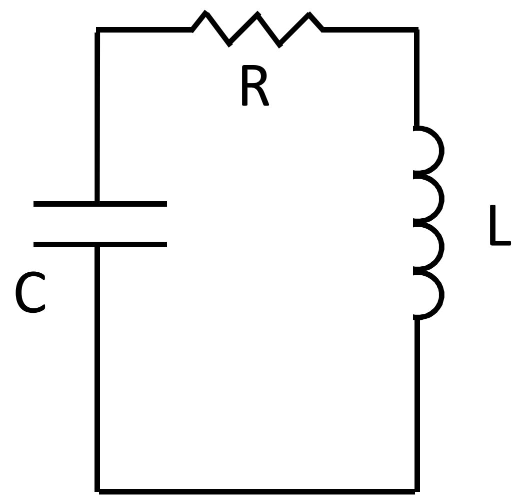
- Using KVL yields:

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$$

- Solutions should oscillate and decay:

$$Q(t) = \sum_k A_k e^{s_k t}$$

RLC Circuits



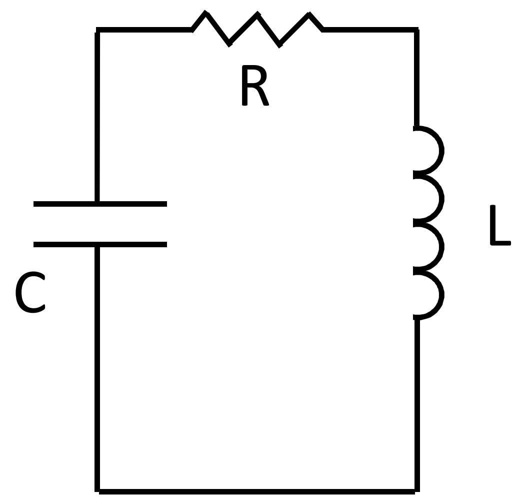
- *Characteristic equation* becomes:

$$s_k{}^2 + \frac{R}{L}s_k + \frac{1}{LC} = 0$$

- Using the quadratic equation:

$$s_k = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

RLC Circuits



- General solution:

$$Q(t) = Ae^{s_+ t} + Be^{s_- t}$$

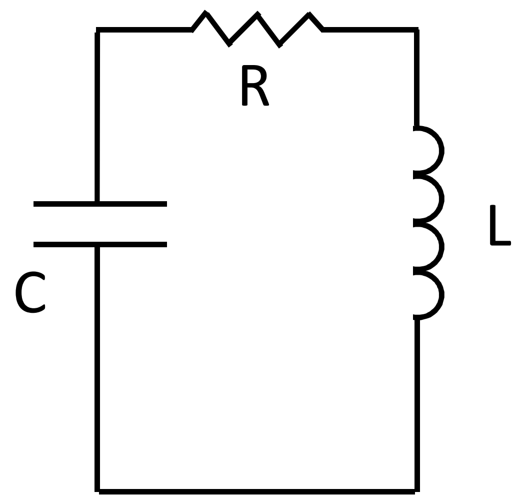
- If $V(0)=V_o$, $I(0)=0$, then:

$$Q(0) = CV_o = A + B$$

$$I(0) = 0 = As_+ + Bs_-$$

$$0 = As_+ + (CV_o - A)s_-$$

RLC Circuits



- Particular solution:

$$A = CV_o s_- / (s_- - s_+)$$

$$B = -CV_o s_+ / (s_- - s_+)$$

$$Q(t) = CV_o \left(\frac{s_- e^{s_+ t} - s_+ e^{s_- t}}{s_- - s_+} \right)$$

General Solutions: RLC Circuits

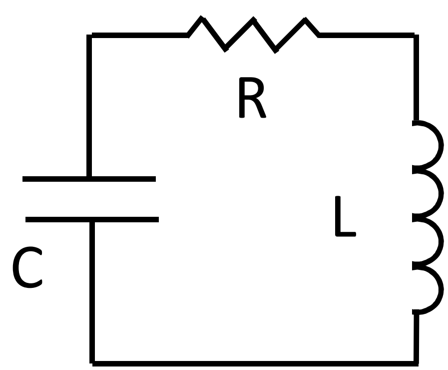
- General solution:

$$Q(t) = A_+ e^{s_+ t} + A_- e^{s_- t}$$

$$s_\pm = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

- If V(0)=V_o, I(0)=0, then:

$$Q(t) = CV_o \left(\frac{s_- e^{s_+ t} - s_+ e^{s_- t}}{s_- - s_+} \right)$$



RLC Circuit

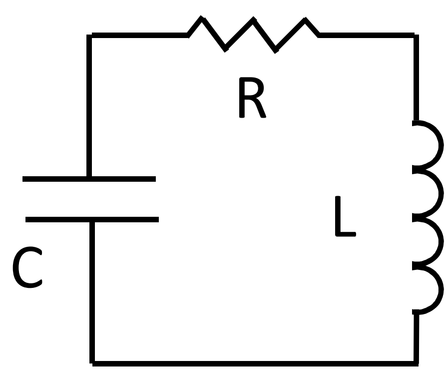
Special Cases: RLC Circuits

- When R=0:

$$s_{\pm} = -\frac{0}{2L} \pm \sqrt{\left(\frac{0}{2L}\right)^2 - \frac{1}{LC}} = \pm\sqrt{-\frac{1}{LC}} = \frac{\pm i}{\sqrt{LC}}$$
$$= \pm i\omega$$

- Solution becomes:

$$Q(t) = A_{+}e^{i\omega t} + A_{-}e^{-i\omega t}$$



RLC Circuit

Special Cases: RLC Circuits

- To capture other regimes, define damping Γ=R/2L, and write solutions:

$$s_{\pm} = -\Gamma \pm \sqrt{\Gamma^2 - \omega_o{}^2}$$

Regime	Value Range	Root type	Behavior
Undamped	$\Gamma = 0$	Pure imaginary	Oscillates forever
Underdamped	$0 < \Gamma < \omega_o$	Complex	Oscillate & decay
Critically damped	$\Gamma = \omega_o$	Pure real	Decay
Overdamped	$\Gamma > \omega_o$	Pure real	Decay

Special Cases: RLC Circuits

- In general, one can define a *quality factor*:

$$Q = \frac{\omega_o}{2\Gamma}$$

Measures the lifetime of an excitation

- Here, Γ=R/2L, and $\omega_o = 1/\sqrt{LC}$, so

$$Q = \frac{1/\sqrt{LC}}{2 \cdot (R/2L)} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

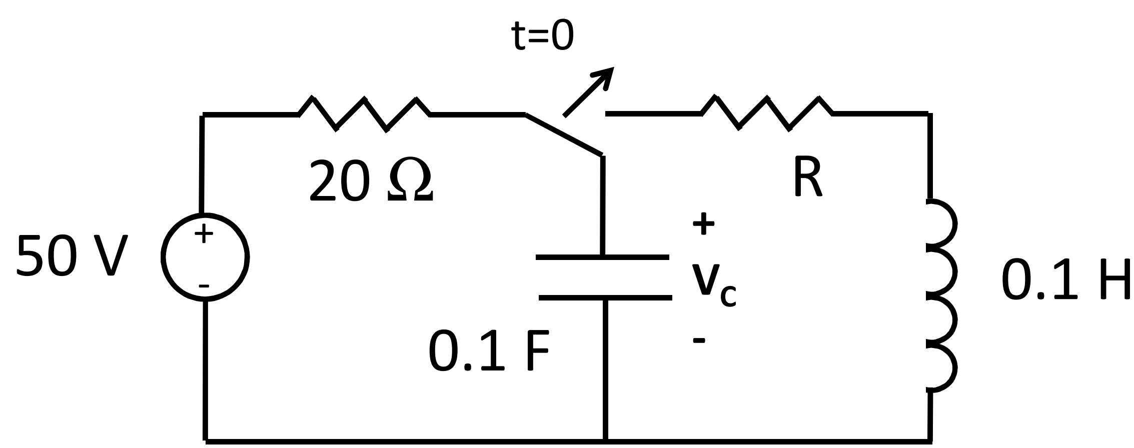
As $\mathrm{R} \to 0$, $\mathrm{Q} \to \infty$, and vice versa

Mathematica Demonstration

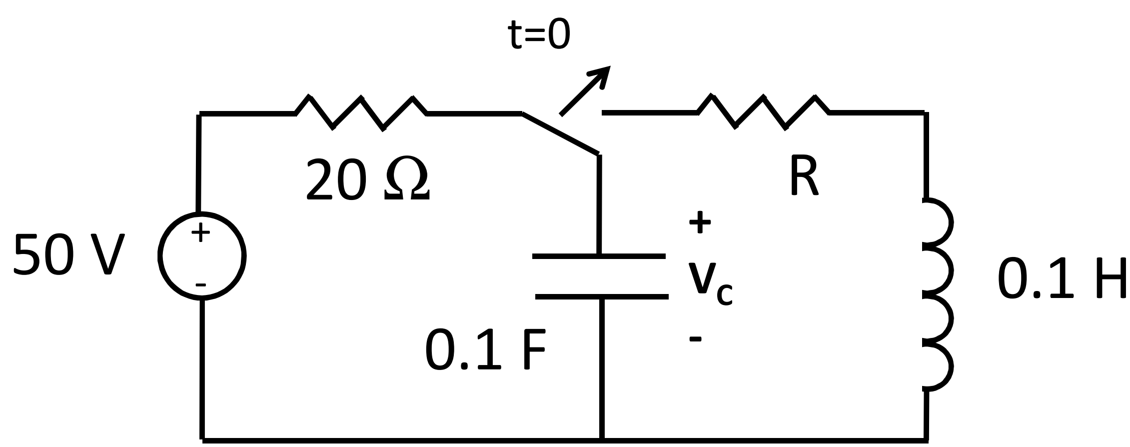
Click to begin

Example 1

- Consider a circuit which charges a capacitor for $t<0$ then switches to an RLC circuit at $t=0$. What is $V_C(t)$ for $R=0$, and 180?



Solution



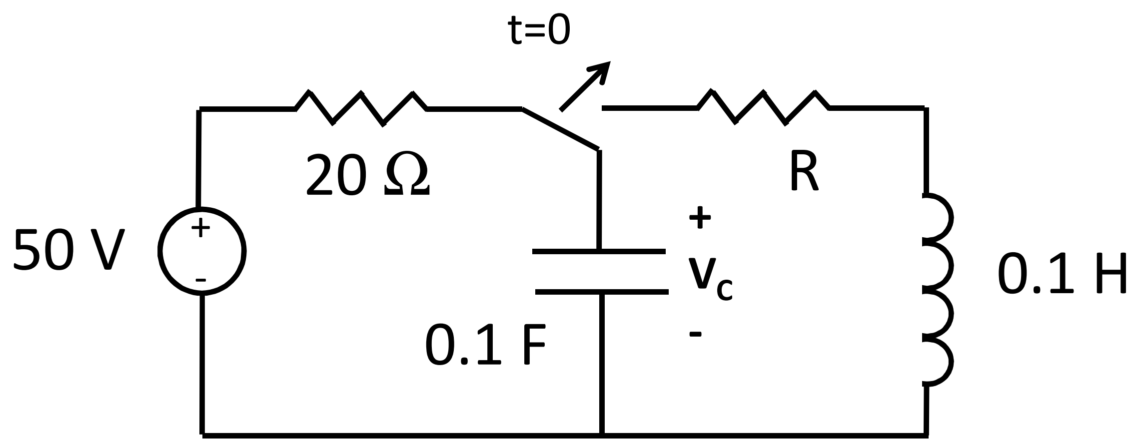
- For t<0: V_C=50 V
- For t>0: I(0)=0, and:

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1 \cdot 0.1}} = 10 \text{ rad/s}$$

$$\Gamma = \frac{R}{2L} = \frac{R}{2 \cdot (0.1\ H)} = 5R$$

$$Q = \frac{\omega}{2\Gamma} = \frac{10}{2 \cdot (5R)} = \frac{1}{R}$$

Solution



- From before:

$$V_C(t) = Q(t)/C = A_+ e^{s_+ t} + A_- e^{s_- t}$$

$$V_C(t) = e^{-\Gamma t}\left(A_+ e^{i\sqrt{\omega^2-\Gamma^2}t} + A_- e^{-i\sqrt{\omega^2-\Gamma^2}t}\right)$$

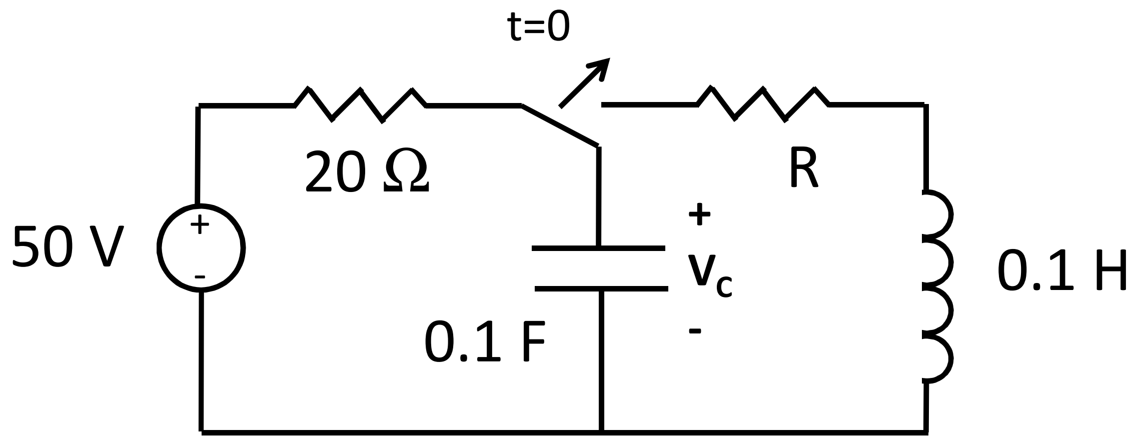
$$V_C(t) = e^{-\Gamma t}\left(A_+ e^{i\omega' t} + A_- e^{-i\omega' t}\right)$$

Matching boundary conditions:

$$A_+ + A_- = V_o$$

$$(-\Gamma + i\omega')A_+ + (-\Gamma - i\omega')A_- = 0$$

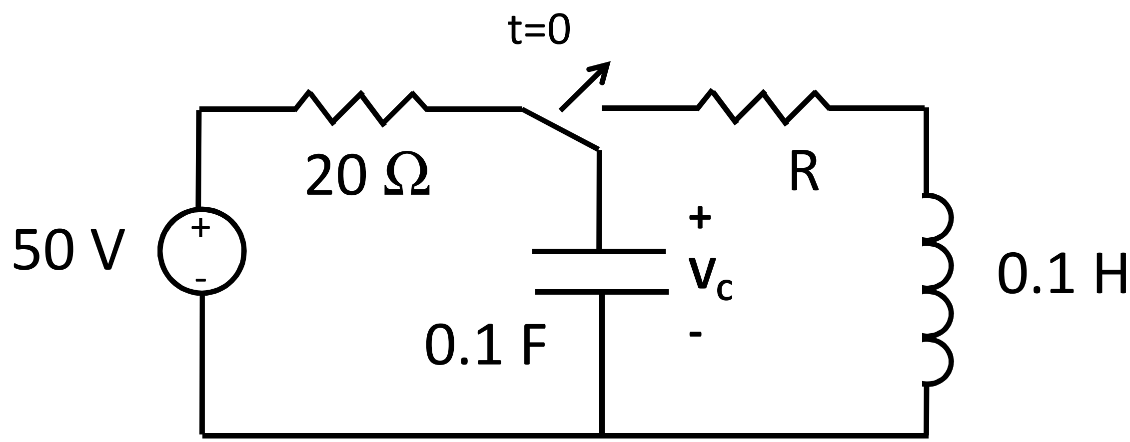
Solution



- Substituting:

$$A_+ + A_- = V_o$$
$$(-\Gamma + i\omega')A_+ + (-\Gamma - i\omega')(V_o - A_+) = 0$$
$$2i\omega' A_+ = (\Gamma + i\omega')V_o$$
$$A_+ = \frac{V_o}{2}\left(1 - i\frac{\Gamma}{\omega'}\right)$$
$$(-\Gamma + i\omega')(V_o - A_-) + (-\Gamma - i\omega')A_- = 0$$
$$-2i\omega' A_- = (\Gamma - i\omega')V_o$$
$$A_- = \frac{V_o}{2}\left(1 + i\frac{\Gamma}{\omega'}\right)$$

Solution



- From before:

$$V_C(t) = \frac{V_o e^{-\Gamma t}}{2}\left[\left(1 - i\frac{\Gamma}{\omega'}\right)e^{i\omega' t} + \left(1 + i\frac{\Gamma}{\omega'}\right)e^{-i\omega' t}\right]$$

$$V_C(t) = V_o e^{-\Gamma t}\left[\cos\omega' t + \left(\frac{\Gamma}{\omega'}\right)\sin\omega' t\right]$$

Homework

- HW #19 solution posted
- HW #20 due today by 4:30 pm in EE 326B
- HW #21 due Wed.: DeCarlo & Lin, Chapter 8:
  - Problem 30
  - Problem 34