ECE 201, Section 3
Lecture 22

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October 17, 2012

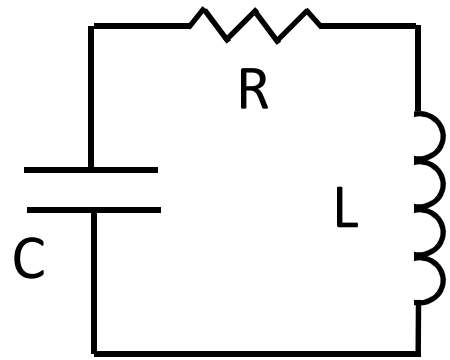
Series RLC Circuits

General solution to RLC circuits:

$$X(t) = A_+ e^{s_+ t} + A_- e^{s_- t}$$

$$s_\pm = -\Gamma \pm \sqrt{\Gamma^2 - {\omega_o}^2}$$

$$\Gamma = R/2L;\ \omega_o = 1/\sqrt{LC}$$

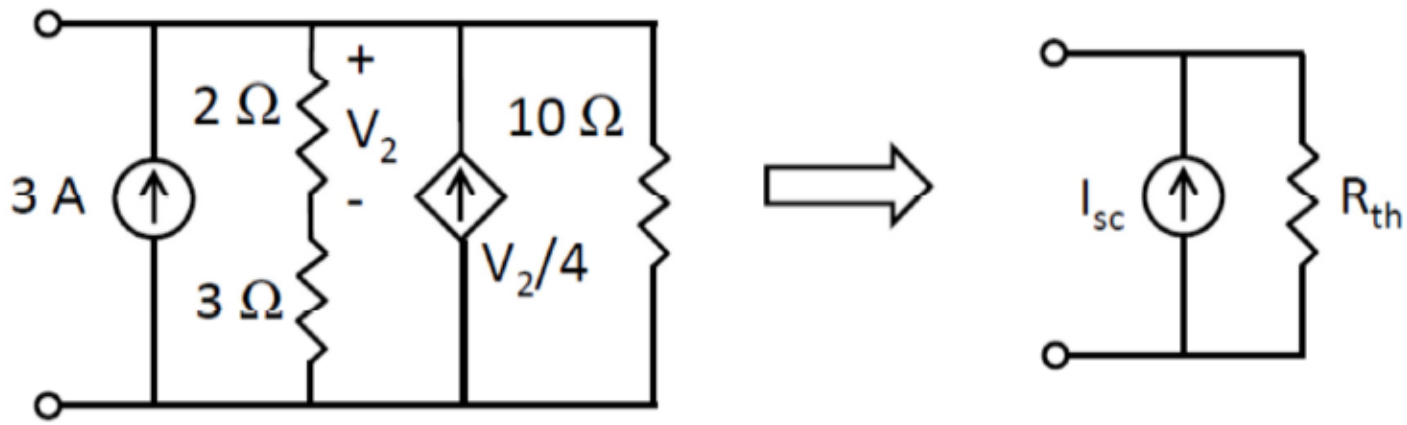


RLC Circuit

Regime	Value Range	Root type	Behavior
Undamped	$\Gamma = 0$	Pure imaginary	Oscillates forever
Underdamped	$0 < \Gamma < \omega_o$	Complex	Oscillate & decay
Critically damped	$\Gamma = \omega_o$	Pure real	Decay
Overdamped	$\Gamma > \omega_o$	Pure real	Decay

Problem 2

2. Given the circuit below on the left, find the Norton equivalent current source and resistance values (as shown on the right):



Answers:

(1) I_{sc}=0 A; R_{th}=10/3 Ω
(2) I_{sc}=3 A; R_{th}=5 Ω
(3) I_{sc}=7/2 A; R_{th}=5 Ω
(4) I_{sc}=9/2 A; R_{th}=10/3 Ω
(5) I_{sc}=9/2 A; R_{th}=10 Ω
(6) I_{sc}=6 A; R_{th}=10/3 Ω
(7) I_{sc}=6 A; R_{th}=10 Ω
(8) None of the above

Problem 3

3. If the following relationship between input current and output voltage is observed for a linear circuit, what are its Norton equivalent current source and resistance values?

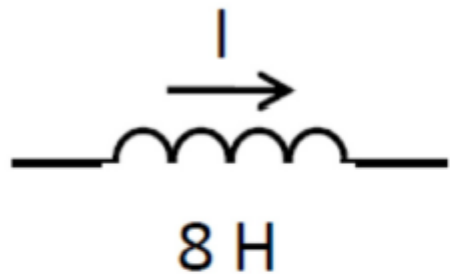
I_A	V_{AB}
2	36
5	48
12	76

Answers:

(1) I_{sc}=0 A; R_{th}=18 Ω
(2) I_{sc}=2 A; R_{th}=9 Ω
(3) I_{sc}=5 A; R_{th}=4.8 Ω
(4) I_{sc}=7 A; R_{th}=4 Ω
(5) I_{sc}=12 A; R_{th}=3 Ω
(6) I_{sc}=14 A; R_{th}=3 Ω
(7) I_{sc}=19 A; R_{th}=2 Ω
(8) None of the above

Problem 6

6. If $I=2^{-t}$ A (i.e., I=1 A at t=0 s, and I=0.25 A at t=2 s), then how much stored energy (in J) is lost from the inductor between t=0 and t=1 s?



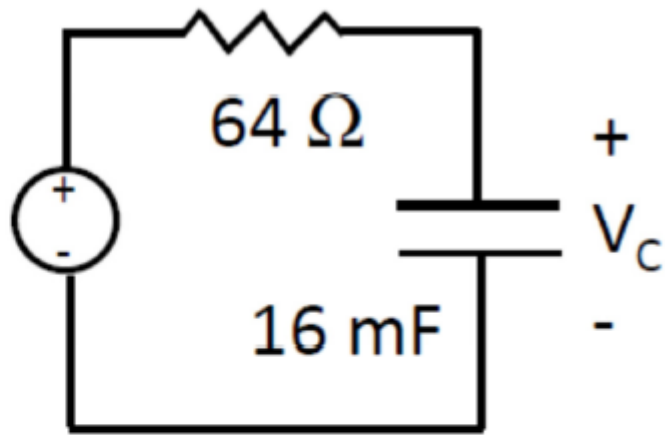
Answers:

(1) 1
(2) 2
(3) 3
(4) 4
(5) 5
(6) 6
(7) 7
(8) None of the above

Problem 9

9. Find the voltage $V_C(t)$ (in V), for t>0. Note: u(t)=1 if t>0, and 0 otherwise.

$$V_{in} = -16u(-t)+32u(t)$$

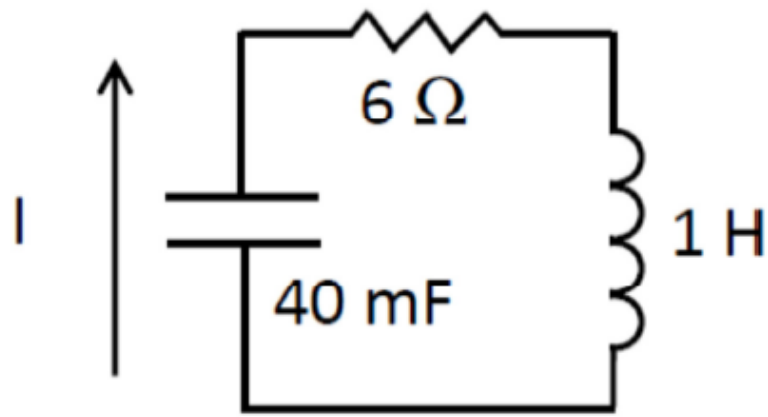


Answers:

(1) $-16e^{-t/1.024}$
(2) $16(1 - e^{-t/1.024})$
(3) $32(1 - e^{-t/1.024})$
(4) $32 - 48e^{-t/1.024}$
(5) $32 - 48e^{-t/2.048}$
(6) $32 - 64e^{t/1.024}$
(7) $64(1- e^{-t/2.048})$
(8) None of the above

Problem 12

12. Find the frequency (in Hz) at which this undriven RLC circuit will oscillate:

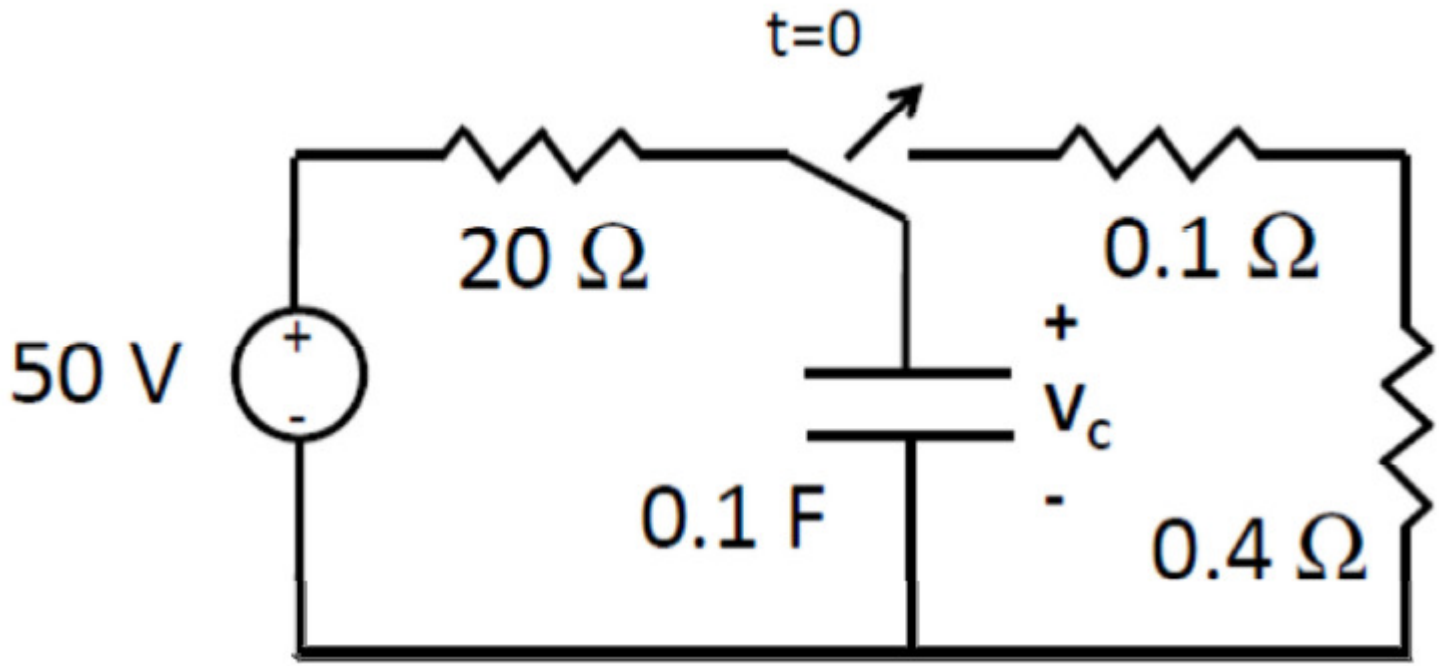


Answers:

(1) 1
(2) 2
(3) 3
(4) 4
(5) 5
(6) 6
(7) 7
(8) None of the above

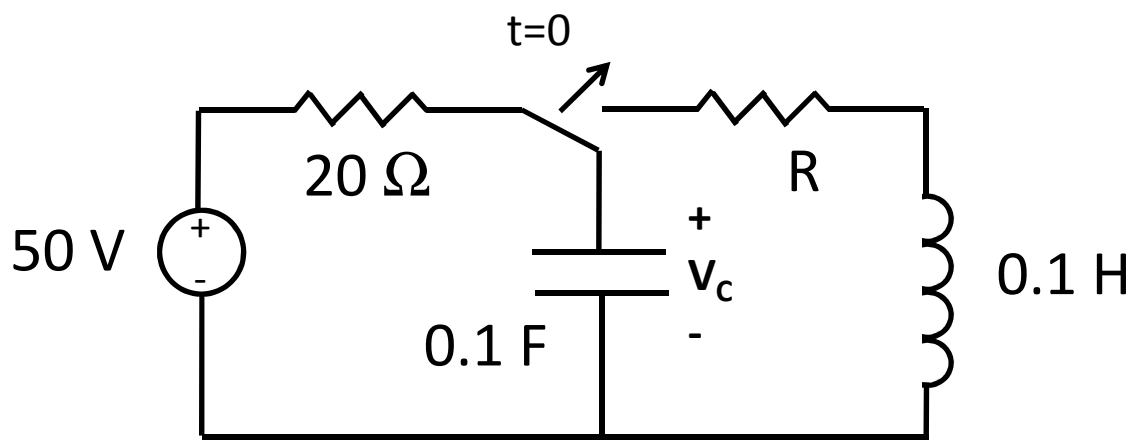
Problem 13

13. Find the voltage across the capacitor V_C (in V) as a function of time after the switch moves from left to right at t=0:

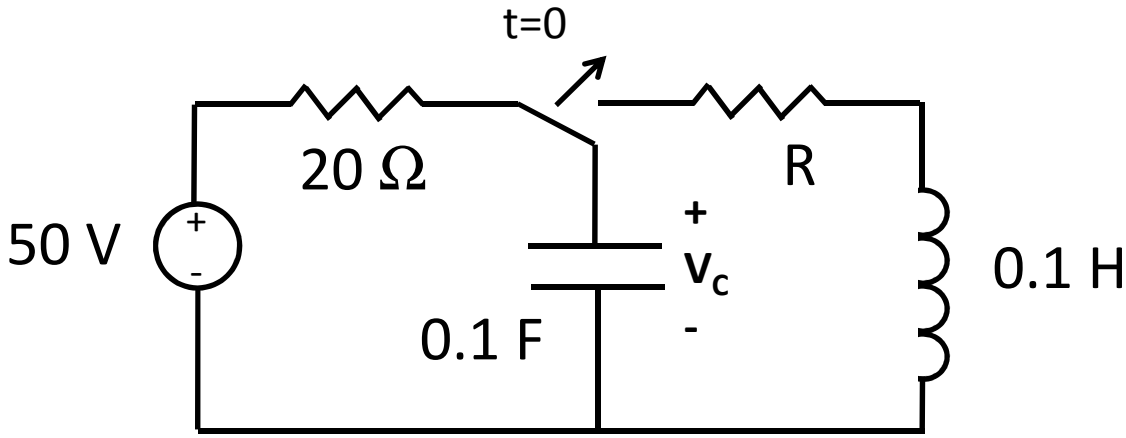


Example 1

- Consider a circuit which charges a capacitor for t<0 then switches to an RLC circuit at t=0. What is $V_C(t)$ for R=0, and 180?



Solution



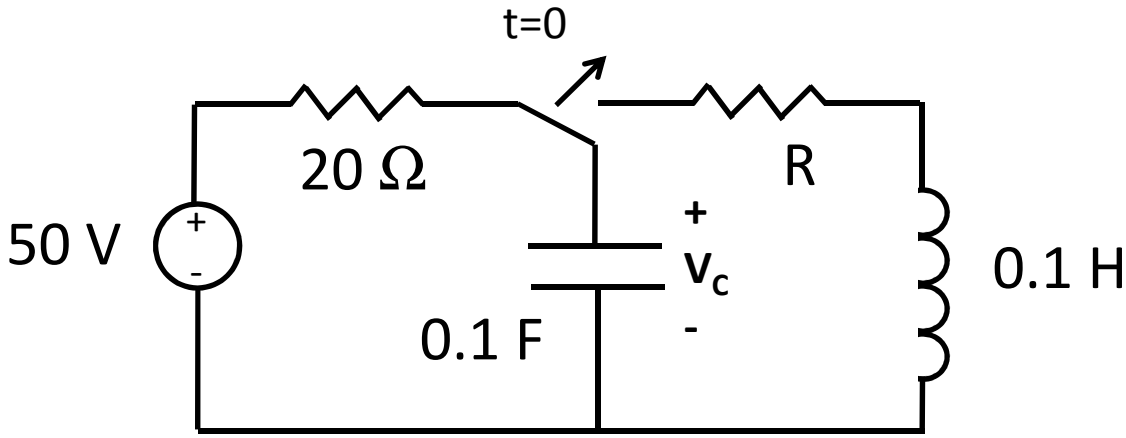
- For t<0: V_C=50 V
- For t>0: I(0)=0, and:

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1 \cdot 0.1}} = 10 \text{ rad/s}$$

$$\Gamma = \frac{R}{2L} = \frac{R}{2 \cdot (0.1\ H)} = 5R$$

$$Q = \frac{\omega}{2\Gamma} = \frac{10}{2 \cdot (5R)} = \frac{1}{R}$$

Solution



- From before:

$$V_C(t) = Q(t)/C = A_+ e^{s_+ t} + A_- e^{s_- t}$$

$$V_C(t) = e^{-\Gamma t}\left(A_+ e^{i\sqrt{\omega^2-\Gamma^2}t} + A_- e^{-i\sqrt{\omega^2-\Gamma^2}t}\right)$$

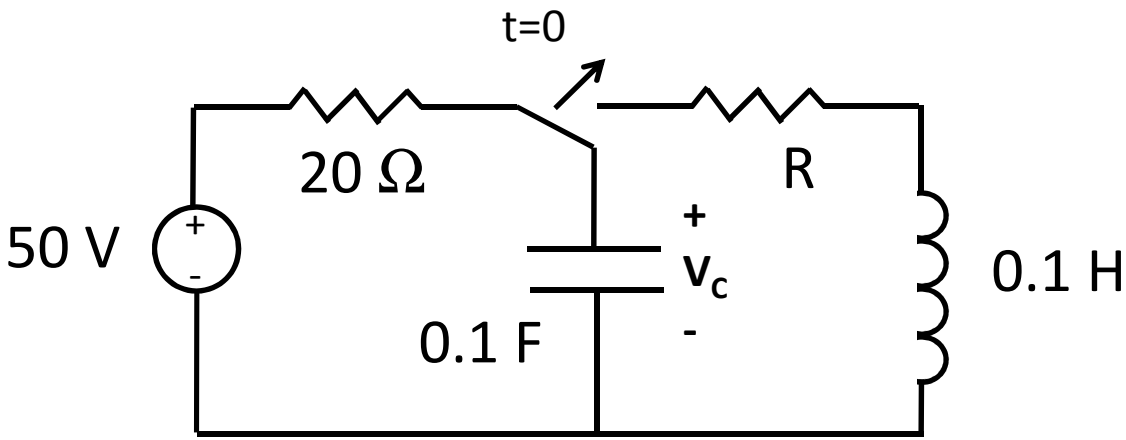
$$V_C(t) = e^{-\Gamma t}\left(A_+ e^{i\omega' t} + A_- e^{-i\omega' t}\right)$$

Matching boundary conditions:

$$A_+ + A_- = V_o$$

$$(-\Gamma + i\omega')A_+ + (-\Gamma - i\omega')A_- = 0$$

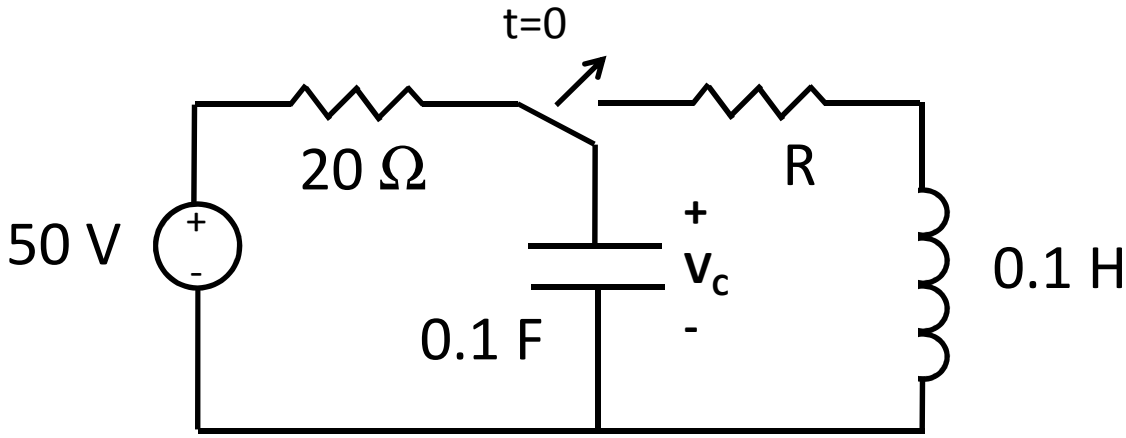
Solution



- Substituting:

$$A_+ + A_- = V_o$$
$$(-\Gamma + i\omega')A_+ + (-\Gamma - i\omega')(V_o - A_+) = 0$$
$$2i\omega' A_+ = (\Gamma + i\omega')V_o$$
$$A_+ = \frac{V_o}{2}\left(1 - i\frac{\Gamma}{\omega'}\right)$$
$$(-\Gamma + i\omega')(V_o - A_-) + (-\Gamma - i\omega')A_- = 0$$
$$-2i\omega' A_- = (\Gamma - i\omega')V_o$$
$$A_- = \frac{V_o}{2}\left(1 + i\frac{\Gamma}{\omega'}\right)$$

Solution



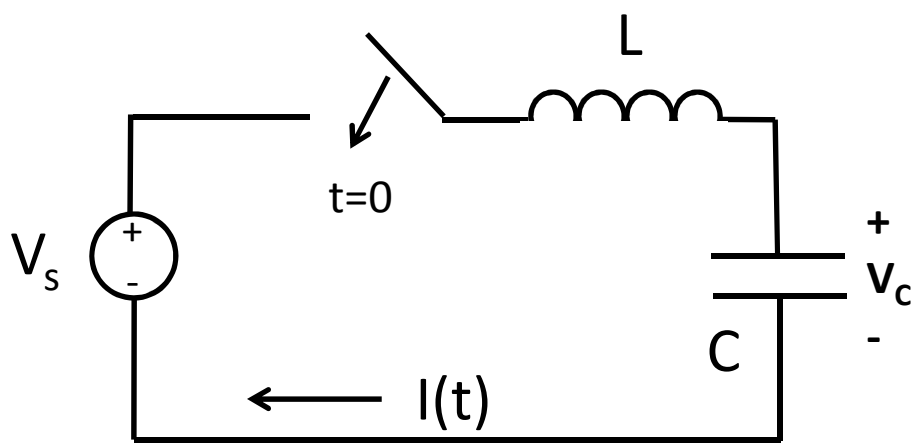
- From before:

$$V_C(t) = \frac{V_o e^{-\Gamma t}}{2}\left[\left(1 - i\frac{\Gamma}{\omega'}\right)e^{i\omega' t} + \left(1 + i\frac{\Gamma}{\omega'}\right)e^{-i\omega' t}\right]$$

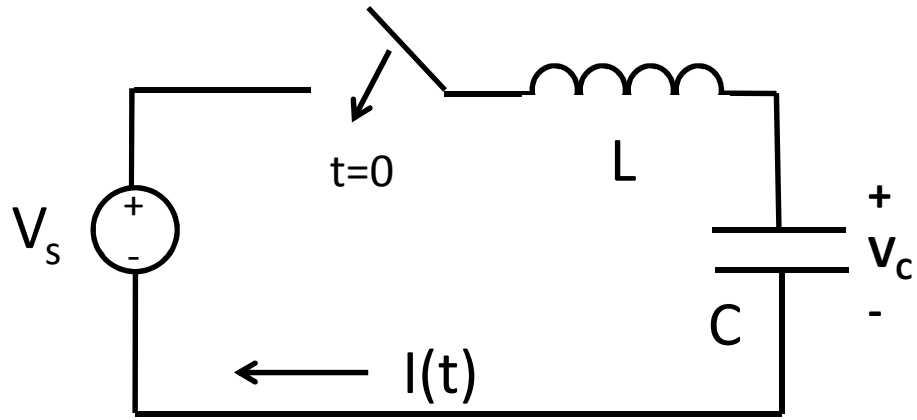
$$V_C(t) = V_o e^{-\Gamma t}\left[\cos\omega' t + \left(\frac{\Gamma}{\omega'}\right)\sin\omega' t\right]$$

Example

- Consider an LC circuit connected to a voltage source at t=0. How will the current and voltage vary with time?



Solution



- Employing our solution:

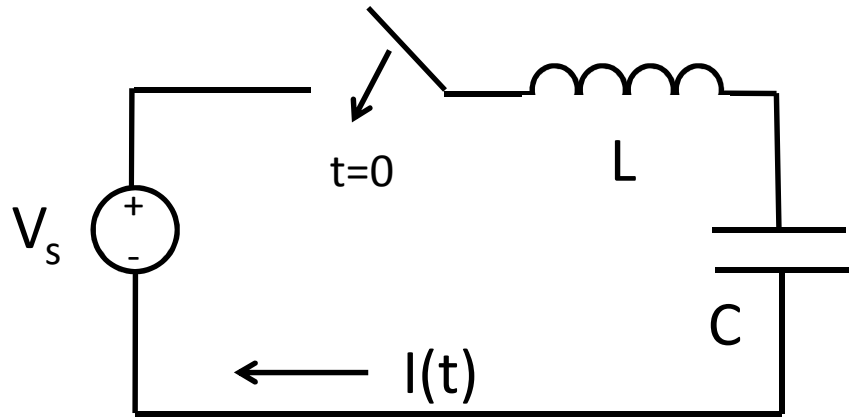
$$I(t) = A_+ e^{s_+ t} + A_- e^{s_- t}$$

$$s_\pm = -\Gamma \pm \sqrt{\Gamma^2 - {\omega_o}^2}$$

$$\Gamma = \frac{R}{2L} = 0; \omega_o = 1/\sqrt{LC}$$

$$s_\pm = \pm i\omega_o$$

Solution



- Employing our solution:

$$I(t) = A_+ e^{i\omega_o t} + A_- e^{-i\omega_o t}$$

$$\frac{dI}{dt} = i\omega_o\left(A_+ e^{i\omega_o t} - A_- e^{-i\omega_o t}\right)$$

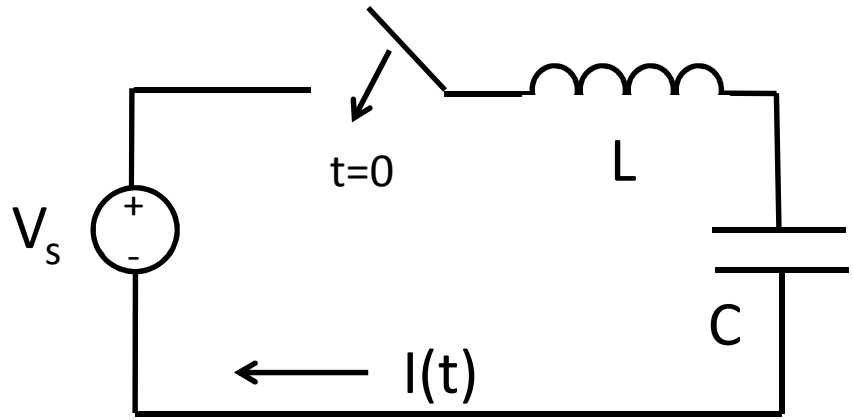
$$I(0) = A_+ + A_- = 0$$

$$V_s = L\frac{dI}{dt}(t=0) = Li\omega_o(A_+ - A_-)$$

$$V_s = 2iL\omega_o A_+ = -2iL\omega_o A_-$$

$$I(t) = \frac{V_s\left(e^{i\omega_o t} - e^{-i\omega_o t}\right)}{2iL\omega_o} = \frac{V_s}{L\omega_o}\sin(\omega_o t)$$

Solution



- Given that:

$$I(t) = \frac{V_s}{L\omega_o}\sin(\omega_o t)$$

We can integrate to obtain:

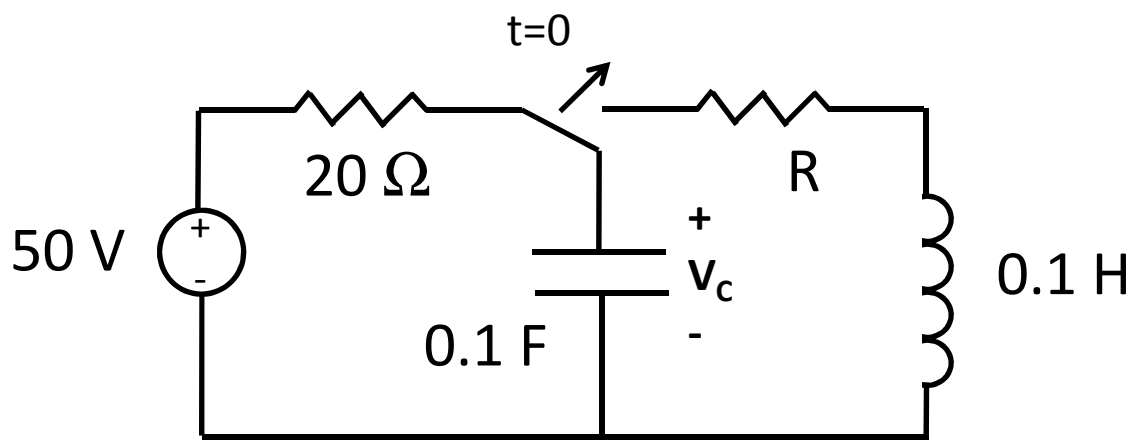
$$V_C(t) = \frac{1}{C}\int_0^t dt' \frac{V_s}{L\omega_o}\sin(\omega_o t') = V_s \int_0^t \omega_o dt' \sin(\omega_o t')$$

$$V_C(t) = -V_s \cos(\omega_o t')\Big|_0^t$$

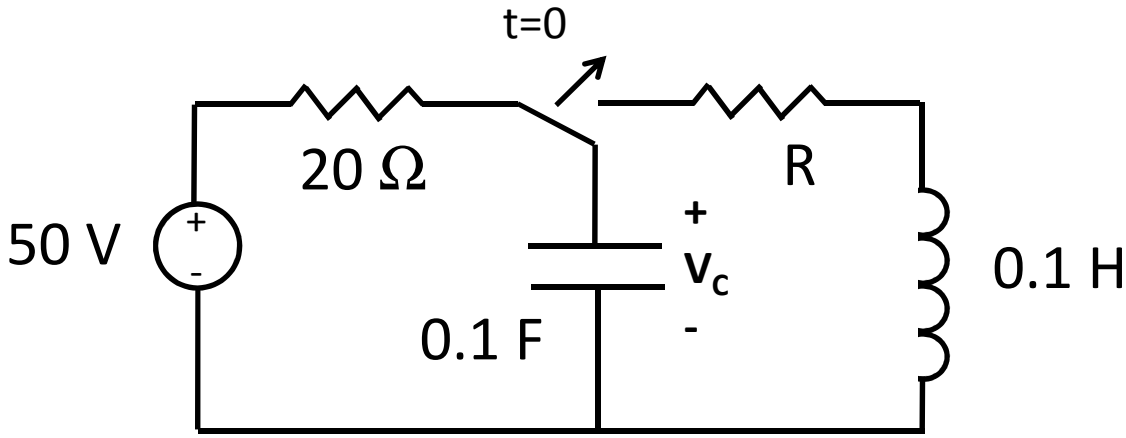
$$V_C(t) = V_s[1 - \cos(\omega_o t)]$$

Example

- Consider a circuit which charges a capacitor for t<0 then switches to an RLC circuit at t=0. What is $V_C(t)$ for R=0, and 180?



Solution



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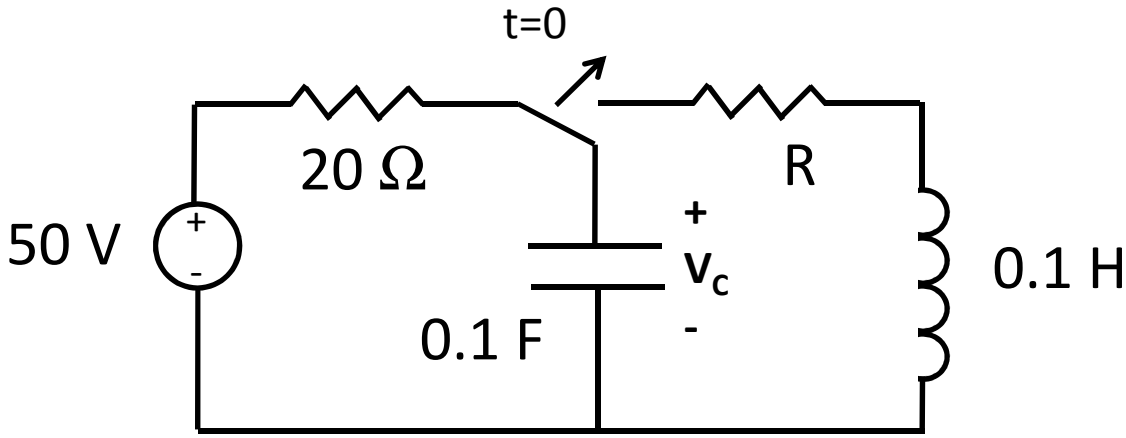
- For t>0: I(0)=0, and:

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1 \cdot 0.1}} = 10 \text{ rad/s}$$

$$\Gamma = \frac{R}{2L} = \frac{R}{2 \cdot (0.1\ H)} = 5R$$

$$Q = \frac{\omega}{2\Gamma} = \frac{10}{2 \cdot (5R)} = \frac{1}{R}$$

Solution



- From before:

$$V_C(t) = Q(t)/C = A_+ e^{s_+ t} + A_- e^{s_- t}$$

$$V_C(t) = e^{-\Gamma t}\left(A_+ e^{i\sqrt{\omega^2-\Gamma^2}t} + A_- e^{-i\sqrt{\omega^2-\Gamma^2}t}\right)$$

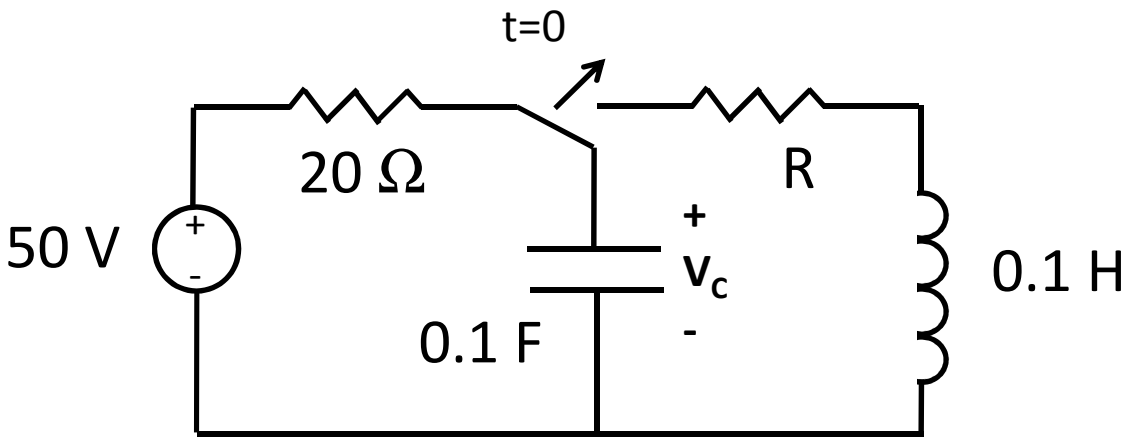
$$V_C(t) = e^{-\Gamma t}\left(A_+ e^{i\omega' t} + A_- e^{-i\omega' t}\right), \text{ where } \omega' = \sqrt{\omega^2 - \Gamma^2}$$

Matching boundary conditions:

$$A_+ + A_- = V_o$$

$$(-\Gamma + i\omega')A_+ + (-\Gamma - i\omega')A_- = 0$$

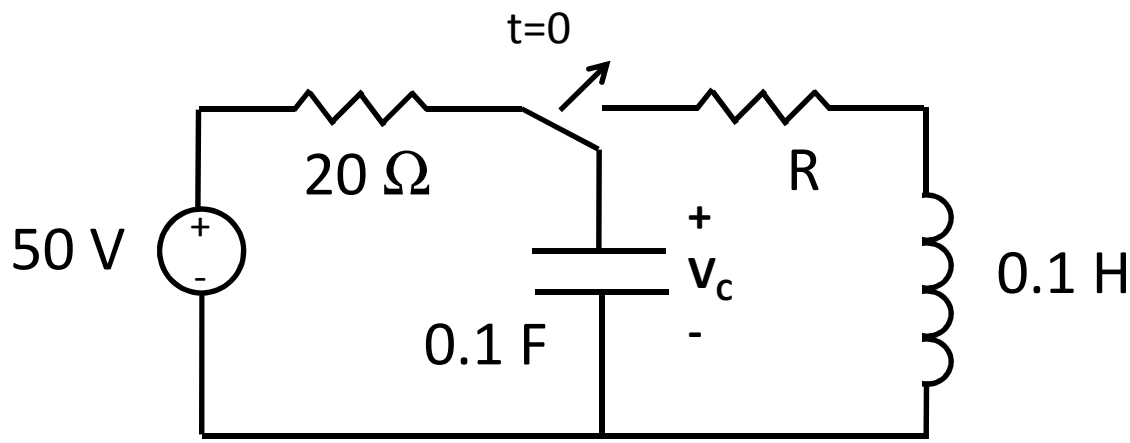
Solution



- Substituting:

$$A_+ + A_- = V_o$$
$$(-\Gamma + i\omega')A_+ + (-\Gamma - i\omega')(V_o - A_+) = 0$$
$$2i\omega' A_+ = (\Gamma + i\omega')V_o$$
$$A_+ = \frac{V_o}{2}\left(1 - i\frac{\Gamma}{\omega'}\right)$$
$$(-\Gamma + i\omega')(V_o - A_-) + (-\Gamma - i\omega')A_- = 0$$
$$-2i\omega' A_- = (\Gamma - i\omega')V_o$$
$$A_- = \frac{V_o}{2}\left(1 + i\frac{\Gamma}{\omega'}\right)$$

Solution



- From before:

$$V_C(t) = \frac{V_o e^{-\Gamma t}}{2}\left[\left(1 - i\frac{\Gamma}{\omega'}\right)e^{i\omega' t} + \left(1 + i\frac{\Gamma}{\omega'}\right)e^{-i\omega' t}\right]$$

$$V_C(t) = V_o e^{-\Gamma t}\left[\cos\omega' t + \left(\frac{\Gamma}{\omega'}\right)\sin\omega' t\right]$$

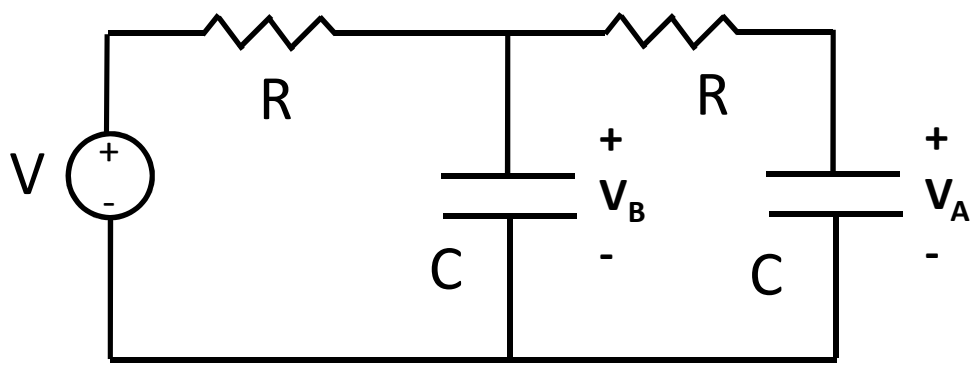
Underdamped for R<2

Critically damped when R=2

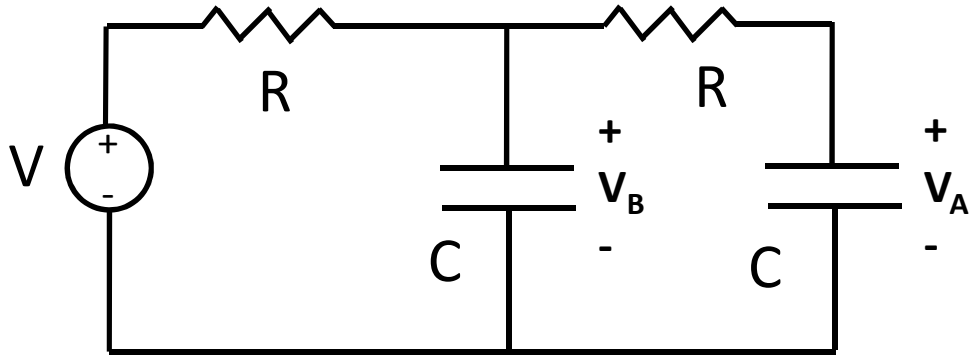
Overdamped when R>2

Example

- What is the equation of motion for the voltage $v_A(t)$ of the right capacitor? What other system does this resemble, and where might you use this design?



Solution

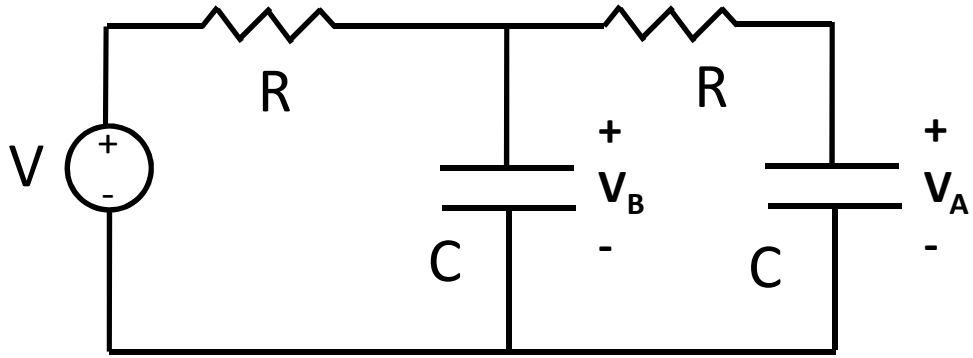


- Taking the derivative of Q=CV and V=IR:

$$\frac{V_B - V_A}{R} = C\frac{dV_A}{dt}$$

$$\frac{V - V_B}{R} + \frac{V_A - V_B}{R} = C\frac{dV_B}{dt}$$

Solution



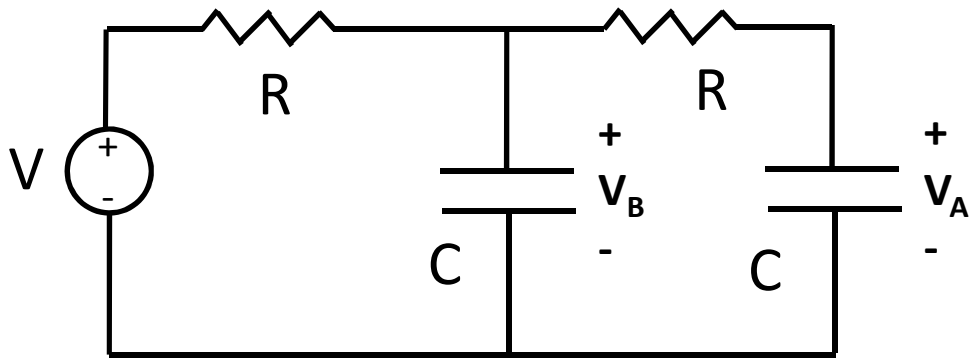
$$V_B - V_A = \tau \frac{dV_A}{dt}$$

$$V + V_A - 2V_B = \tau \frac{dV_B}{dt}$$

Substituting, $V + V_A - 2\left(V_A + \tau \frac{dV_A}{dt}\right) = \tau \frac{dV_B}{dt}$

Time derivative yields, $\frac{dV_B}{dt} - \frac{dV_A}{dt} = \tau \frac{d^2V_A}{dt^2}$

Solution



$$V + V_A - 2V_A - 2\tau\frac{dV_A}{dt} = \tau\left(\tau\frac{d^2V_A}{dt^2} + \frac{dV_A}{dt}\right)$$

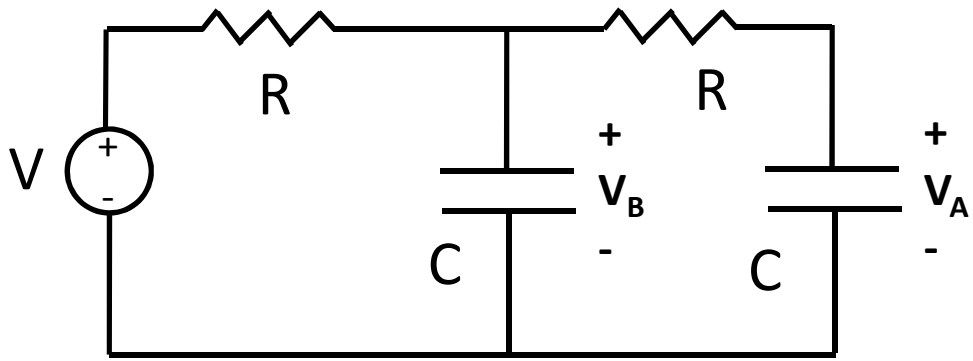
$$V - V_A - 3\tau\frac{dV_A}{dt} = \tau^2\frac{d^2V_A}{dt^2}$$

$$\tau^2\frac{d^2V_A}{dt^2} + 3\tau\frac{dV_A}{dt} + V_A = V$$

Two first order ODEs became one second order ODE:

"Conservation of misery"

Solution



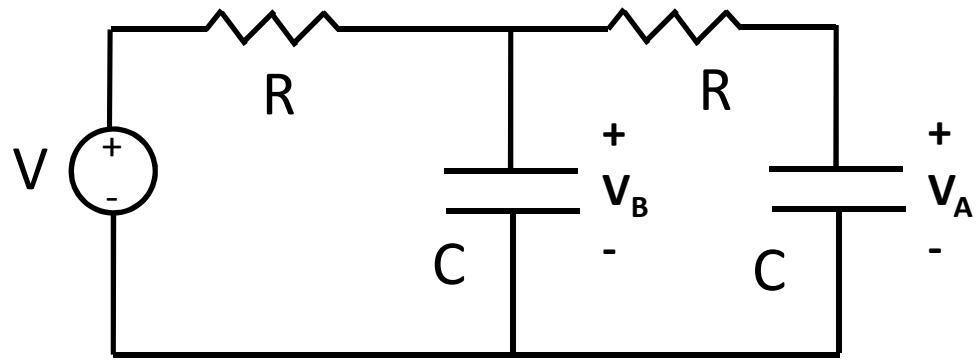
$$\tau^2 \frac{d^2 V_A}{dt^2} + 3\tau \frac{dV_A}{dt} + V_A = V$$

Cf. $\frac{d^2 Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC} Q = 0$

$$\frac{d^2 V_A}{dt^2} + \frac{3}{\tau}\frac{dV_A}{dt} + \frac{1}{\tau^2} V_A = \frac{V}{\tau^2}$$

Thus, $\Gamma = \frac{R}{2L} = \frac{3}{2\tau}$ and $\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\tau}$

Solution



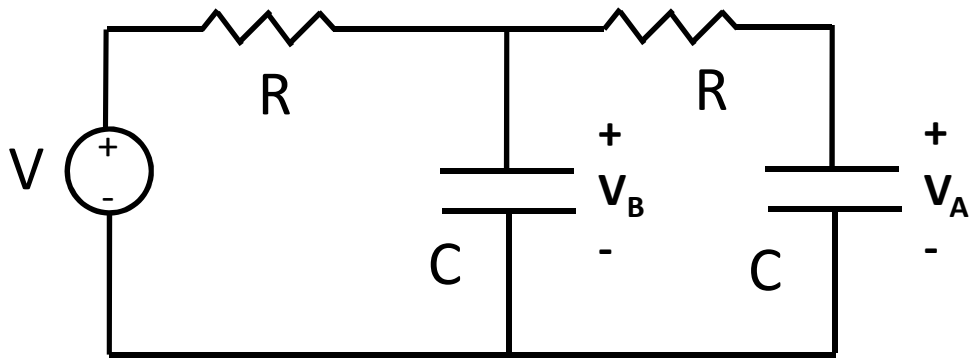
Since $\Gamma = \frac{3}{2\tau}$ and $\omega_o = \frac{1}{\tau}$, $\Gamma > \omega_o$

$$s_{\pm} = -\Gamma \pm \sqrt{\Gamma^2 - {\omega_o}^2}$$

$$= -\frac{3}{2\tau} \pm \sqrt{\left(\frac{3}{2\tau}\right)^2 - \left(\frac{1}{\tau}\right)^2} = -\frac{3}{2\tau} \pm \frac{\sqrt{5}}{2\tau}$$

Overdamped circuit

Solution



- This circuit offers an alternative oscillator
- Application: timing on a microchip!
- Hard to fit inductors on microchips, but relatively easy to put resistors and capacitors on
- Real clock signals generated in a conceptually related fashion, but more sophisticated

Homework

- HW #21 due today by 4:30 pm in EE 326B
- HW #22 due Fri.: DeCarlo & Lin, Chapter 9:
  - Problem 8
  - For the circuit in the previous problem, find the instantaneous power absorbed by each of the elements. What is the sum of these powers?