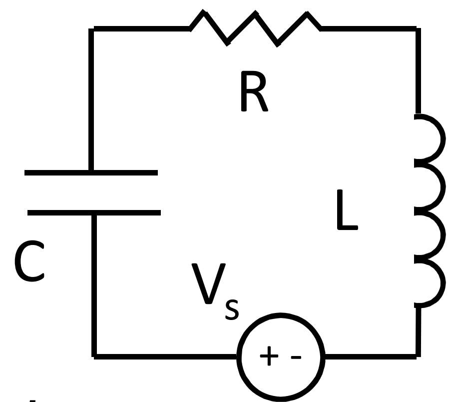
ECE 201, Section 3
Lecture 23

Prof. Peter Bermel

October 19, 2012

Driven Series RLC Circuits

From KVL:



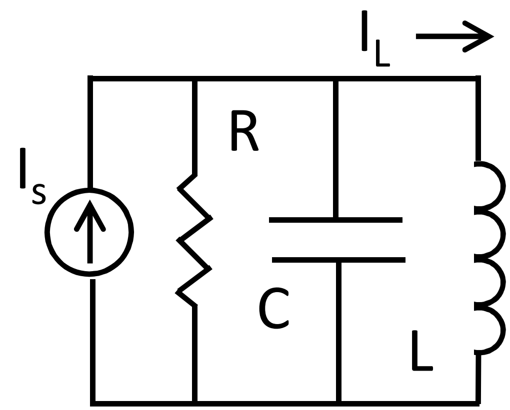
$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{V_s}{L}$$

$$\frac{d^2V_C}{dt^2} + 2\Gamma\frac{dV_C}{dt} + \omega_o^2 V_C = \omega_o^2 V_s$$

$$\Gamma = R/2L;\ \omega_o = 1/\sqrt{LC};\ \omega' = \sqrt{\omega_o^2 - \Gamma^2} = -i\Gamma'$$

Regime	Range	Solution	Behavior
Under-damped	$\Gamma < \omega_o$	$V_C(t) = e^{-\Gamma t}\cos(\omega' t + \phi) + V_s$	Oscillate & decay
Critically damped	$\Gamma = \omega_o$	$V_C(t) = e^{-\Gamma t}(A_1 + A_2 t) + V_s$	Decay
Over-damped	$\Gamma > \omega_o$	$V_C(t) = e^{-\Gamma t}\left(A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t}\right) + V_s$	Decay

Driven Parallel RLC Circuits



From KCL:

$$\frac{d^2 I_L}{dt^2} + \frac{1}{RC}\frac{dI_L}{dt} + \frac{1}{LC} I_L = \frac{I_S}{LC}$$

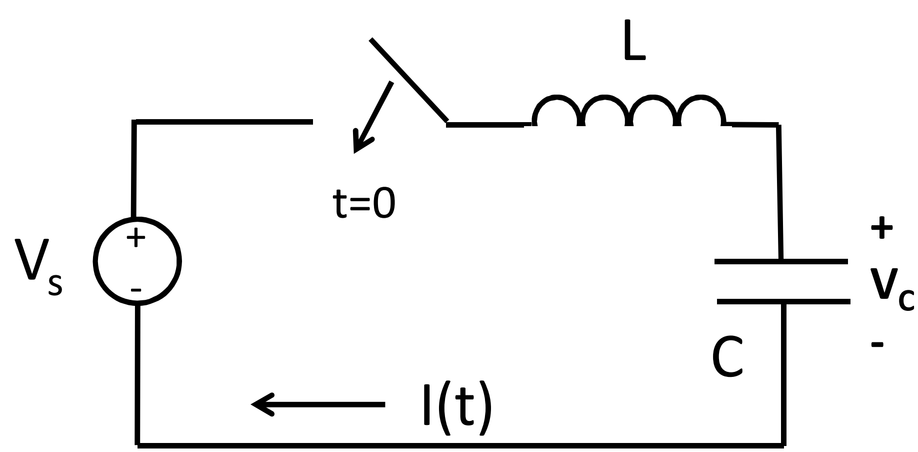
$$\frac{d^2 I_L}{dt^2} + 2\Gamma\frac{dI_L}{dt} + \omega_o^2 I_L = \omega_o^2 I_s$$

$$\Gamma = 1/(2RC);\ \omega_o = 1/\sqrt{LC};\ \omega' = \sqrt{\omega_o^2 - \Gamma^2} = -i\Gamma'$$

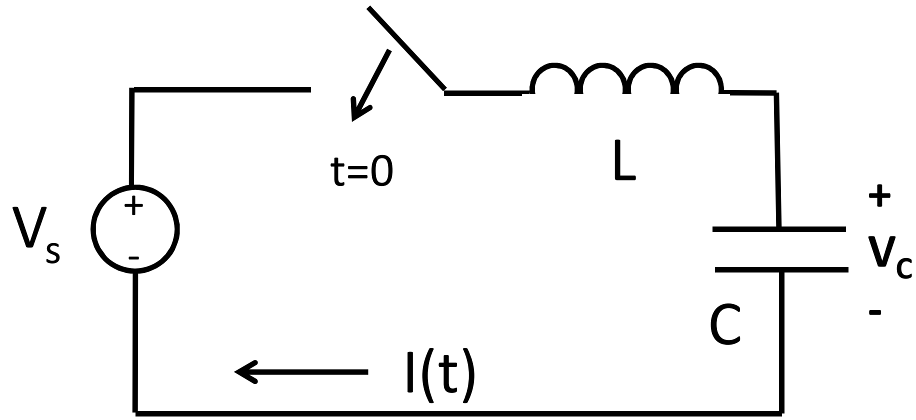
Regime	Range	Solution	Behavior
Under-damped	$\Gamma < \omega_o$	$I_L(t) = e^{-\Gamma t}\cos(\omega' t + \phi) + I_s$	Oscillate & decay
Critically damped	$\Gamma = \omega_o$	$I_L(t) = e^{-\Gamma t}(A_1 + A_2 t) + I_s$	Decay
Over-damped	$\Gamma > \omega_o$	$I_L(t) = e^{-\Gamma t}\left(A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t}\right) + I_s$	Decay

Example

- Consider an LC circuit connected to a voltage source at t=0. How will the current and voltage vary with time?



Solution



- Employing our solution:

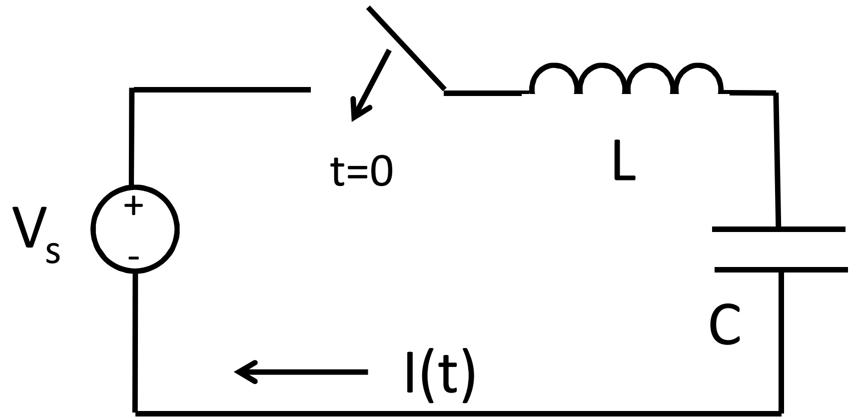
$$I(t) = A_+ e^{s_+ t} + A_- e^{s_- t}$$

$$s_\pm = -\Gamma \pm \sqrt{\Gamma^2 - {\omega_o}^2}$$

$$\Gamma = \frac{R}{2L} = 0;\, \omega_o = 1/\sqrt{LC}$$

$$s_\pm = \pm i\omega_o$$

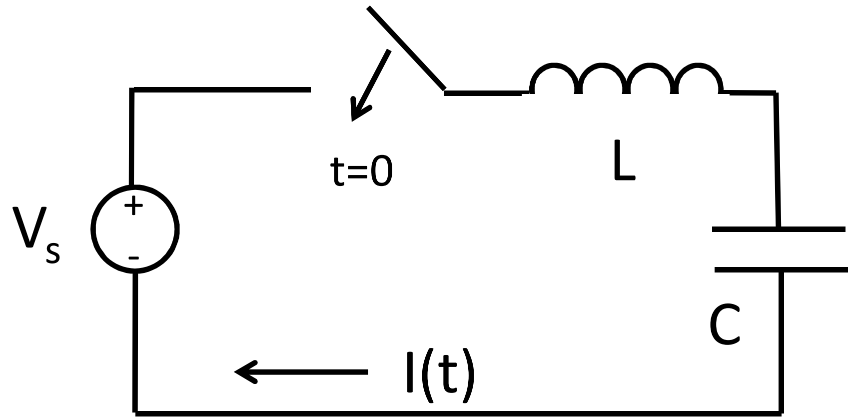
Solution



- Employing our solution:

$$I(t) = A_+ e^{i\omega_o t} + A_- e^{-i\omega_o t}$$
$$\frac{dI}{dt} = i\omega_o\left(A_+ e^{i\omega_o t} - A_- e^{-i\omega_o t}\right)$$
$$I(0) = A_+ + A_- = 0$$
$$V_s = L\frac{dI}{dt}(t=0) = Li\omega_o(A_+ - A_-)$$
$$V_s = 2iL\omega_o A_+ = -2iL\omega_o A_-$$
$$I(t) = \frac{V_s\left(e^{i\omega_o t} - e^{-i\omega_o t}\right)}{2iL\omega_o} = \frac{V_s}{L\omega_o}\sin(\omega_o t)$$

Solution



- Given that:

$$I(t) = \frac{V_s}{L\omega_o}\sin(\omega_o t)$$

We can integrate to obtain:

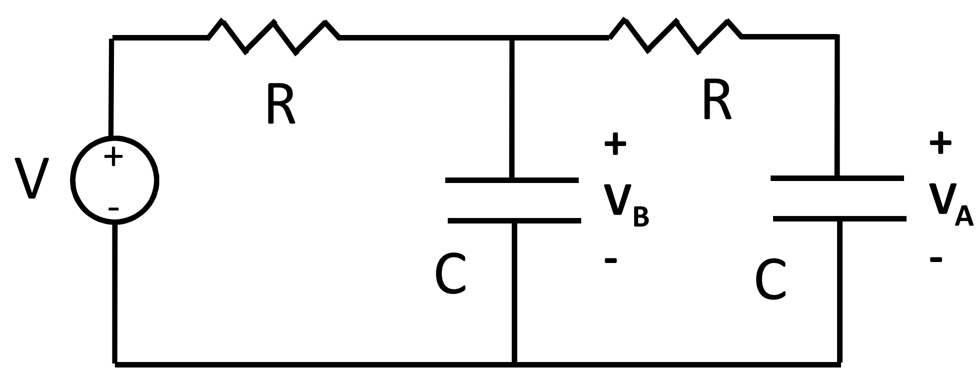
$$V_C(t) = \frac{1}{C}\int_0^t dt' \frac{V_s}{L\omega_o}\sin(\omega_o t') = V_s\int_0^t \omega_o dt' \sin(\omega_o t')$$

$$V_C(t) = -V_s\cos(\omega_o t')\Big|_0^t$$

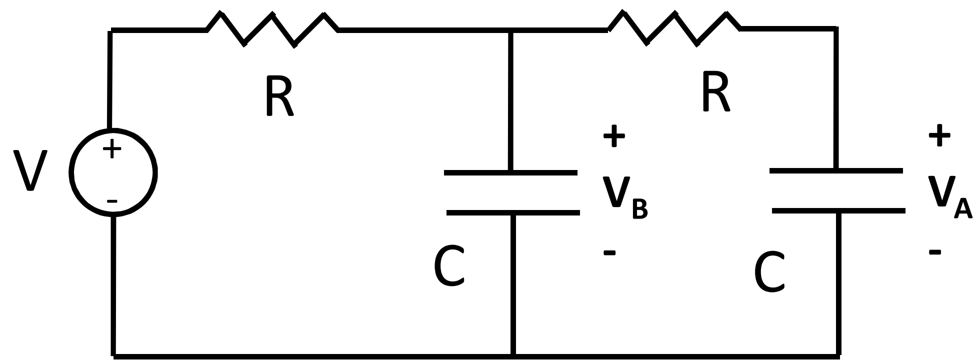
$$V_C(t) = V_s[1-\cos(\omega_o t)]$$

Example

- What is the equation of motion for the voltage $v_A(t)$ of the right capacitor? What other system does this resemble, and where might you use this design?



Solution

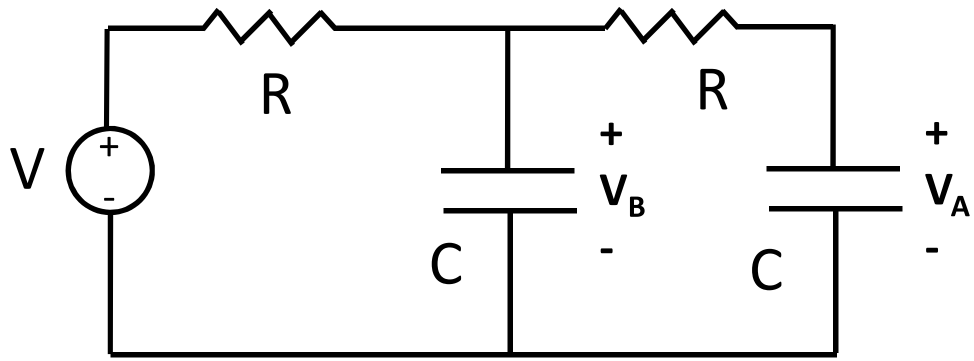


- Taking the derivative of Q=CV and V=IR:

$$\frac{V_B - V_A}{R} = C\frac{dV_A}{dt}$$

$$\frac{V - V_B}{R} + \frac{V_A - V_B}{R} = C\frac{dV_B}{dt}$$

Solution



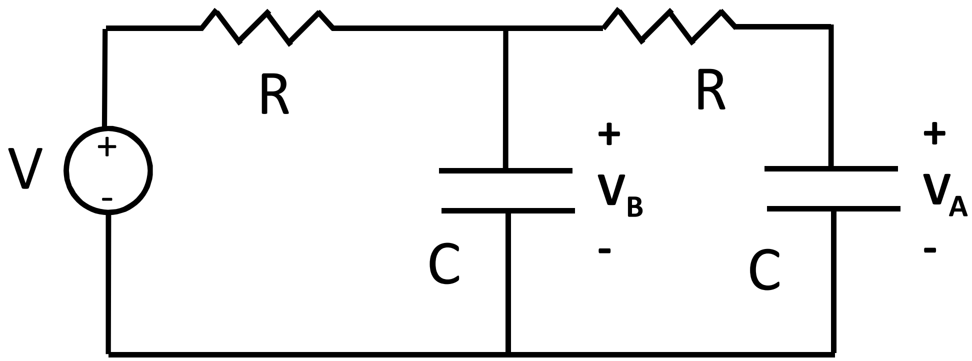
$$V_B - V_A = \tau \frac{dV_A}{dt}$$

$$V + V_A - 2V_B = \tau \frac{dV_B}{dt}$$

Substituting, $V + V_A - 2\left(V_A + \tau \frac{dV_A}{dt}\right) = \tau \frac{dV_B}{dt}$

Time derivative yields, $\frac{dV_B}{dt} - \frac{dV_A}{dt} = \tau \frac{d^2V_A}{dt^2}$

Solution



$$V + V_A - 2V_A - 2\tau\frac{dV_A}{dt} = \tau\left(\tau\frac{d^2V_A}{dt^2} + \frac{dV_A}{dt}\right)$$

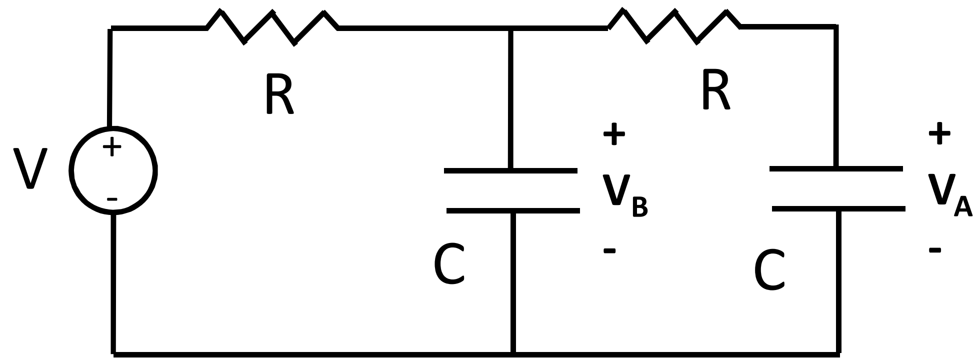
$$V - V_A - 3\tau\frac{dV_A}{dt} = \tau^2\frac{d^2V_A}{dt^2}$$

$$\tau^2\frac{d^2V_A}{dt^2} + 3\tau\frac{dV_A}{dt} + V_A = V$$

Two first order ODEs became one second order ODE:

“Conservation of misery”

Solution



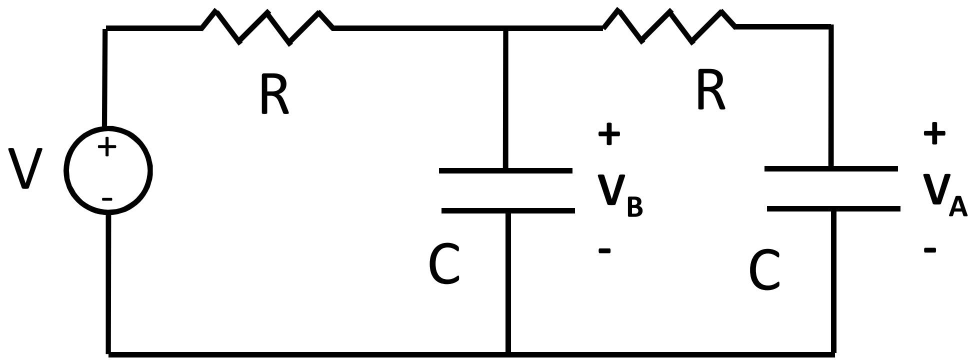
$$\tau^2 \frac{d^2V_A}{dt^2} + 3\tau \frac{dV_A}{dt} + V_A = V$$

Cf. $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$

$$\frac{d^2V_A}{dt^2} + \frac{3}{\tau}\frac{dV_A}{dt} + \frac{1}{\tau^2}V_A = \frac{V}{\tau^2}$$

Thus, $\Gamma = \frac{R}{2L} = \frac{3}{2\tau}$ and $\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\tau}$

Solution



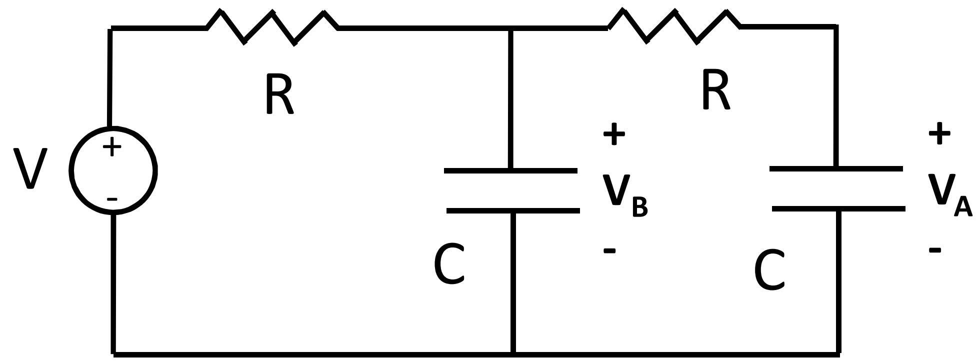
Since $\Gamma = \frac{3}{2\tau}$ and $\omega_o = \frac{1}{\tau}$, $\Gamma > \omega_o$

$$s_\pm = -\Gamma \pm \sqrt{\Gamma^2 - {\omega_o}^2}$$

$$= -\frac{3}{2\tau} \pm \sqrt{\left(\frac{3}{2\tau}\right)^2 - \left(\frac{1}{\tau}\right)^2} = -\frac{3}{2\tau} \pm \frac{\sqrt{5}}{2\tau}$$

Overdamped circuit

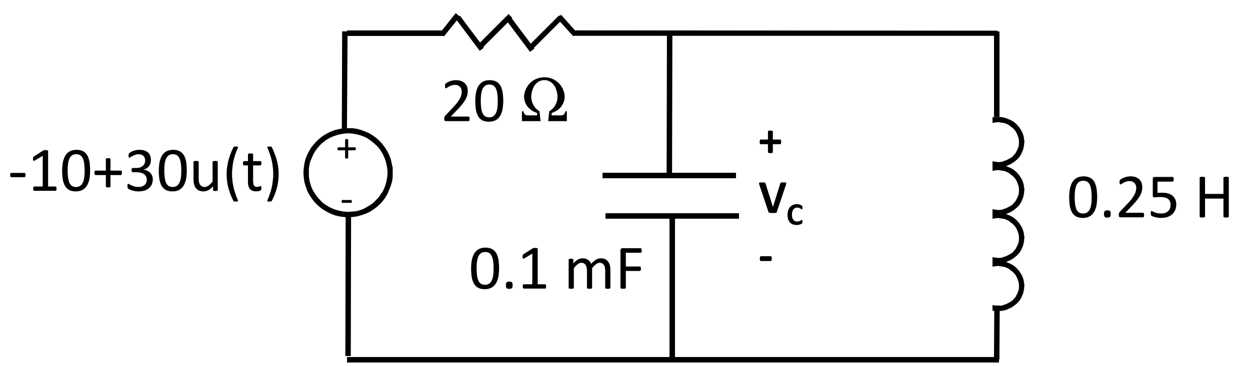
Solution



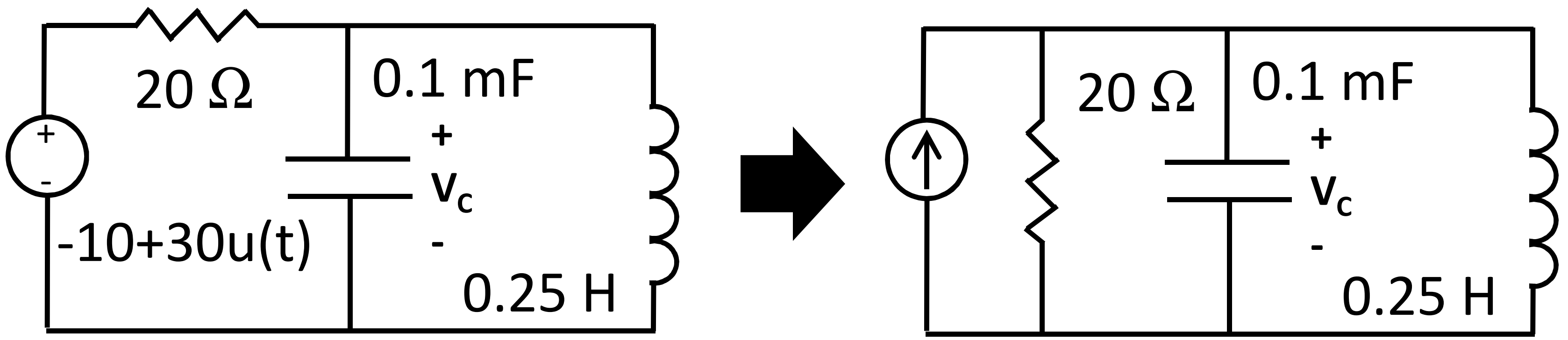
- This circuit offers an alternative oscillator
- Application: timing on a microchip!
- Hard to fit inductors on microchips, but relatively easy to put resistors and capacitors on
- Real clock signals generated in a conceptually related fashion, but more sophisticated

Example

- Consider a circuit in which input voltage switches from -10 V to +20 V at t=0 (i.e., $V = -10 + 30u(t)$). What is V_C(t) at all times?



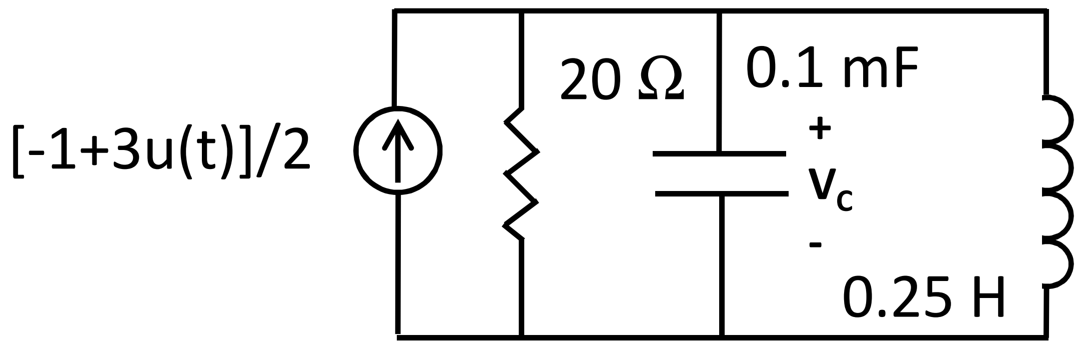
Solution



- Use source transformation to recover original parallel RLC circuit problem:

$$I_s = \frac{[-10 + 30u(t)]}{20} = -\frac{1}{2} + \frac{3}{2}u(t)$$

Solution



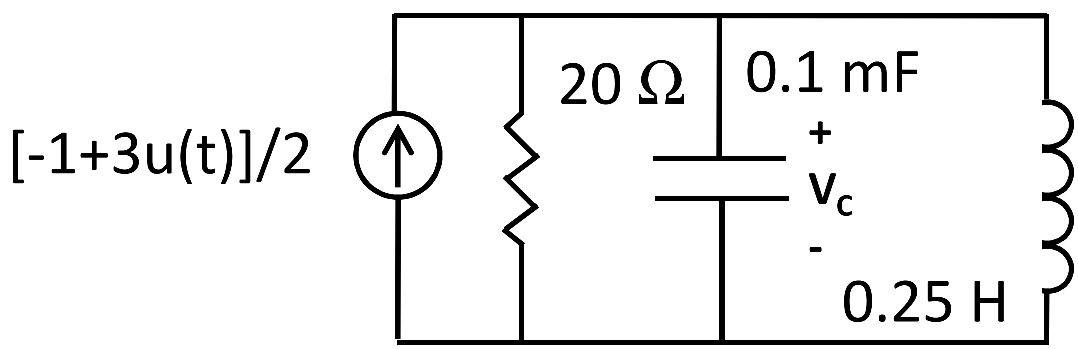
- From earlier definitions:

$$\Gamma = \frac{1}{2RC} = \frac{1}{2 \cdot 20 \cdot 10^{-4}} = 250$$

$$\omega_o = \frac{1}{\sqrt{0.25 \cdot 10^{-4}}} = 200$$

- Circuit is overdamped

Solution



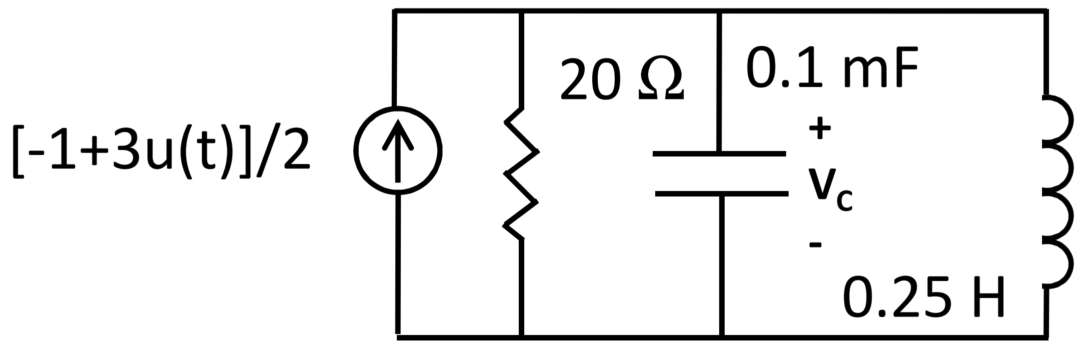
- For t<0, inductor looks like short and $V_C(t<0)=0$
- For t>0, non-zero voltage temporarily possible; start by solving for I_L in overdamped regime:

$$I_L(t) = e^{-\Gamma t}\left(A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t}\right) + I_s$$

Here, $\Gamma = 250$, and

$$\Gamma' = \sqrt{\Gamma^2 - \omega_o^2} = \sqrt{250^2 - 200^2} = 150$$

Solution



- Given that:

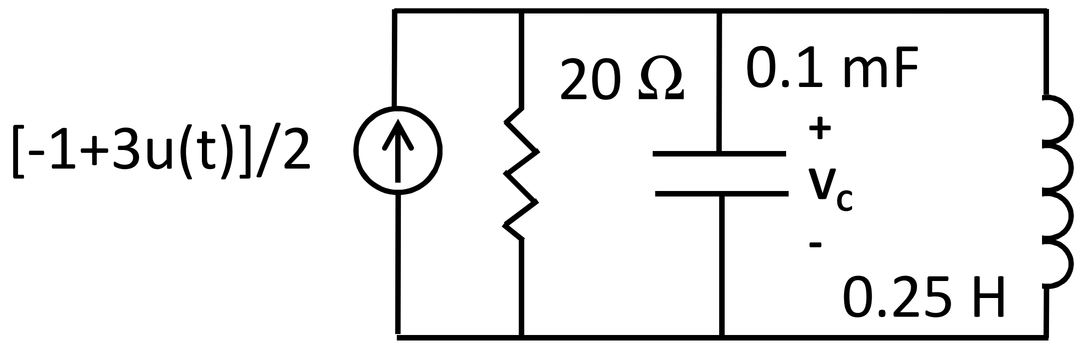
$$I_L(t \geq 0) = e^{-250t}(A_+e^{150t} + A_-e^{-150t}) + 1$$
$$= A_+e^{-100t} + A_-e^{-400t} + 1$$

- Since $I_L(0)=-1/2$, and $V_C(0)=0$:

$$-\frac{1}{2} = A_+ + A_- + 1 \rightarrow A_+ = -\frac{3}{2} - A_-$$
$$0.25(-100A_+ - 400A_-) = 0 \rightarrow A_+ = -4A_-$$

- Thus, $A_- = 1/2$, and $A_+ = -2$

Solution

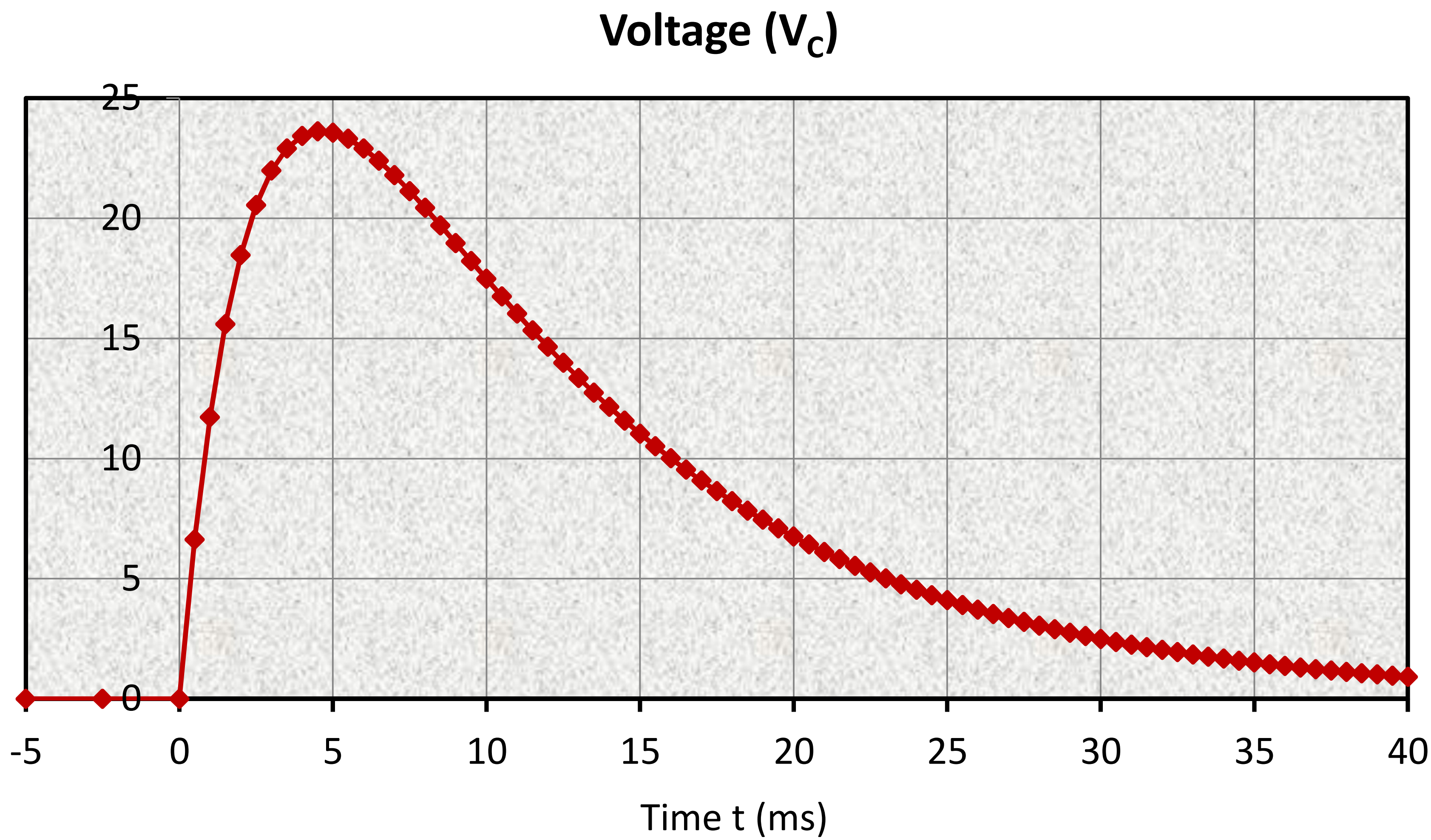


$$I_L(t \geq 0) = -2e^{-100t} + \frac{1}{2}e^{-400t} + 1$$

- Since circuit elements are in parallel:

$$V_C(t \geq 0) = L\frac{dI_L}{dt}$$
$$= 0.25\left(-2(-100)e^{-100t} + \frac{1}{2}(-400)e^{-400t}\right)$$
$$= 50(e^{-100t} - e^{-400t})$$

Solution



Homework

- HW #22 due today by 4:30 pm in EE 326B
- HW #23 due Mon.: DeCarlo & Lin, Chapter 9:
  - Problem 11(a) [Correction: the second $i_L(0^-)$ should be $i_L(0^+)$]
  - Problem 18
  - Problem 27