ECE 201, Section 3
Lecture 24

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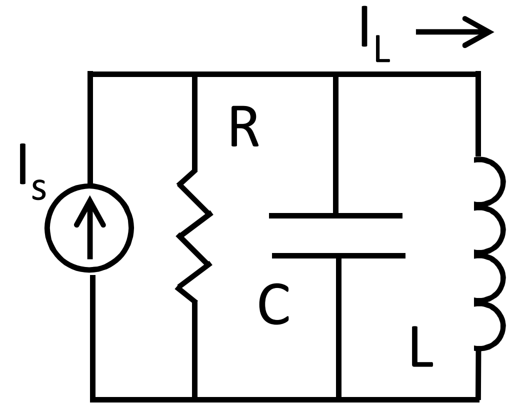
October 22, 2012

Driven Parallel RLC Circuits



From KCL:

$$\frac{d^2 I_L}{dt^2} + \frac{1}{RC}\frac{dI_L}{dt} + \frac{1}{LC} I_L = \frac{I_S}{LC}$$

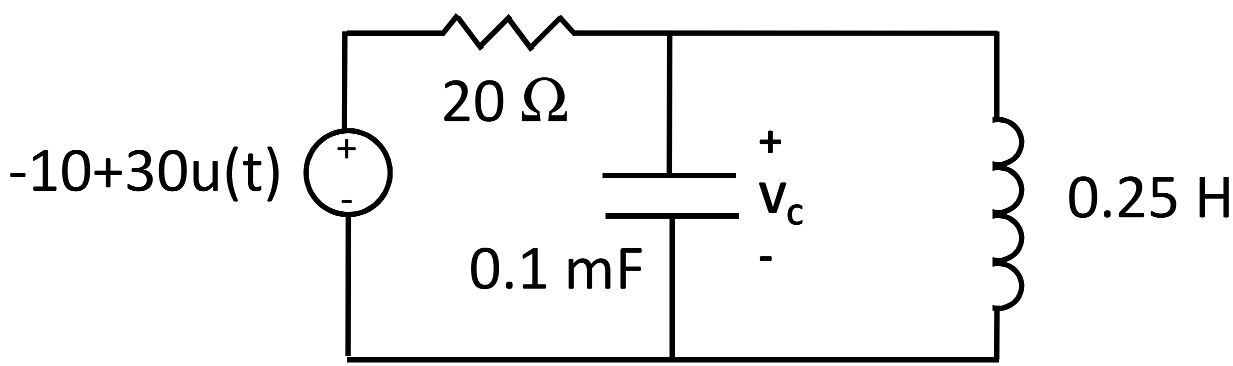
$$\frac{d^2 I_L}{dt^2} + 2\Gamma\frac{dI_L}{dt} + \omega_o^2 I_L = \omega_o^2 I_s$$

$$\Gamma = 1/(2RC);\ \omega_o = 1/\sqrt{LC};\ \omega' = \sqrt{\omega_o^2 - \Gamma^2} = -i\Gamma'$$

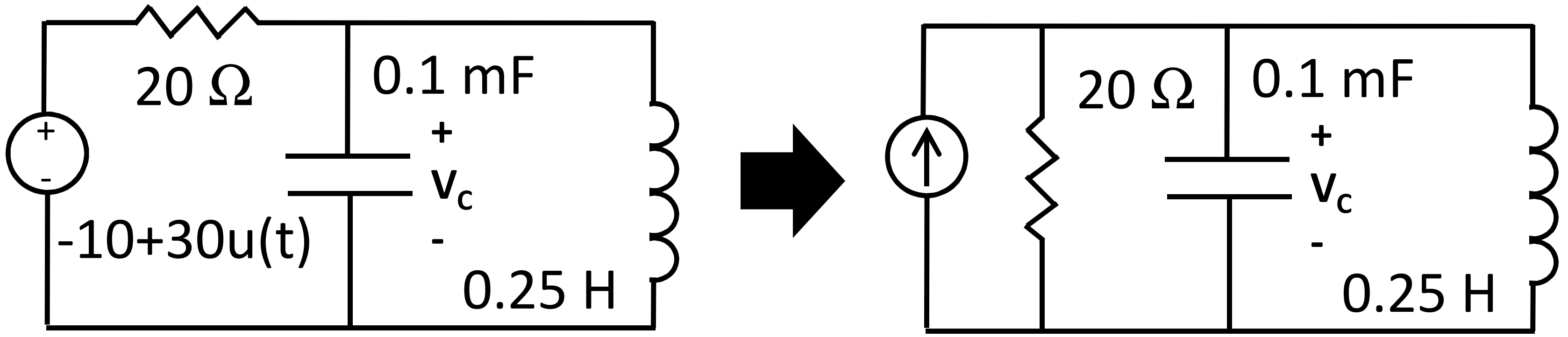
Regime	Range	Solution	Behavior
Under-damped	$\Gamma < \omega_o$	$I_L(t) = I_o e^{-\Gamma t}\cos(\omega' t + \phi) + I_s$	Oscillate & decay
Critically damped	$\Gamma = \omega_o$	$I_L(t) = e^{-\Gamma t}(A_1 + A_2 t) + I_s$	Decay
Over-damped	$\Gamma > \omega_o$	$I_L(t) = e^{-\Gamma t}\left(A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t}\right) + I_s$	Decay

Example (continued from last time)

- Consider a circuit in which input voltage switches from -10 V to +20 V at t=0 (i.e., $V = -10 + 30u(t)$). What is $V_C(t)$ at all times?



Solution



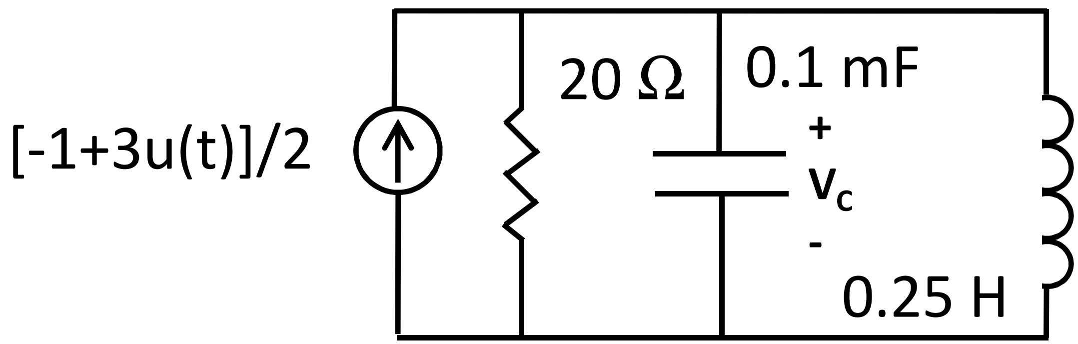
- Recall:

$$I_s = -\frac{1}{2} + \frac{3}{2}u(t)$$

$$\Gamma = \frac{1}{2RC} = \frac{1}{2 \cdot 20 \cdot 10^{-4}} = 250$$

$$\omega_o = \frac{1}{\sqrt{0.25 \cdot 10^{-4}}} = 200$$

Solution



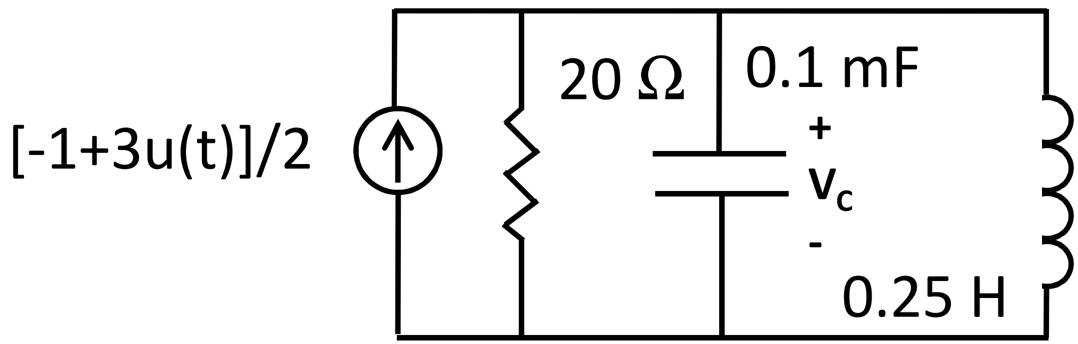
- For t<0, inductor looks like short and $V_C(t<0)=0$
- For t>0, non-zero voltage temporarily possible; start by solving for I_L in overdamped regime:

$$I_L(t) = e^{-\Gamma t}\left(A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t}\right) + I_s$$

Here, $\Gamma = 250$, and

$$\Gamma' = \sqrt{\Gamma^2 - \omega_o^2} = \sqrt{250^2 - 200^2} = 150$$

Solution



- Given that:

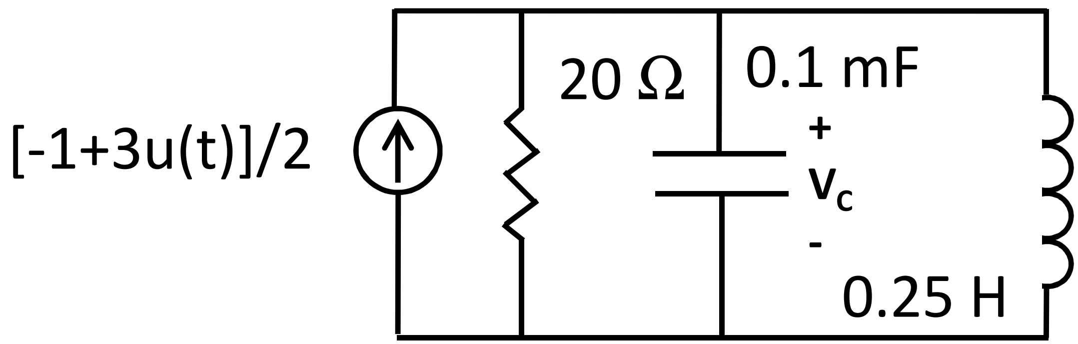
$$I_L(t \geq 0) = e^{-250t}(A_+ e^{150t} + A_- e^{-150t}) + 1$$
$$= A_+ e^{-100t} + A_- e^{-400t} + 1$$

- Since $I_L(0)$=-1/2, and $V_C(0)$=0:

$$-\frac{1}{2} = A_+ + A_- + 1 \rightarrow A_+ = -\frac{3}{2} - A_-$$
$$0.25(-100A_+ - 400A_-) = 0 \rightarrow A_+ = -4A_-$$

- Thus, $A_- = 1/2$, and $A_+ = -2$

Solution



$$I_L(t \geq 0) = -2e^{-100t} + \frac{1}{2}e^{-400t} + 1$$

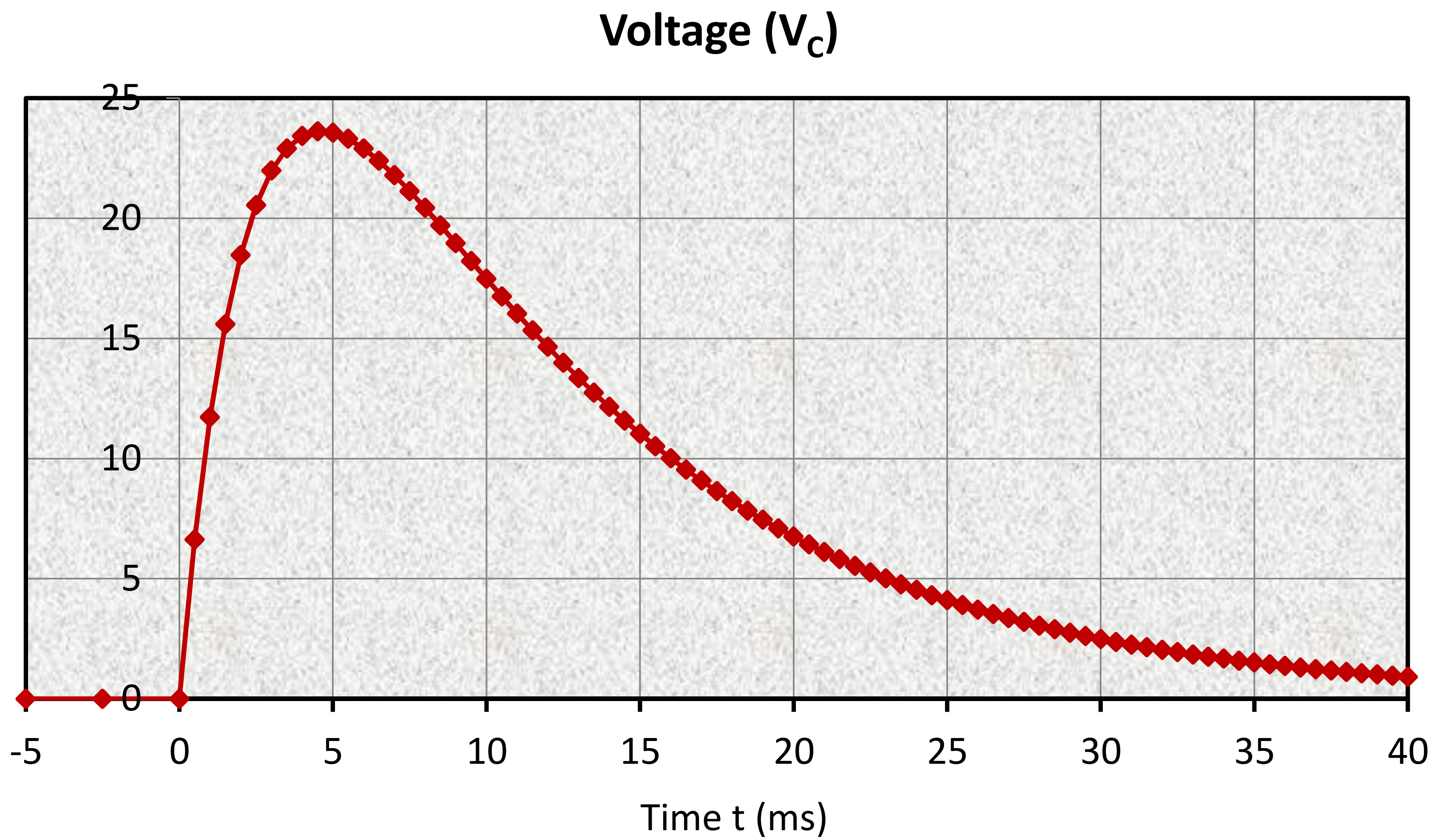
- Since circuit elements are in parallel:

$$V_C(t \geq 0) = L\frac{dI_L}{dt}$$

$$= 0.25\left(-2(-100)e^{-100t} + \frac{1}{2}(-400)e^{-400t}\right)$$

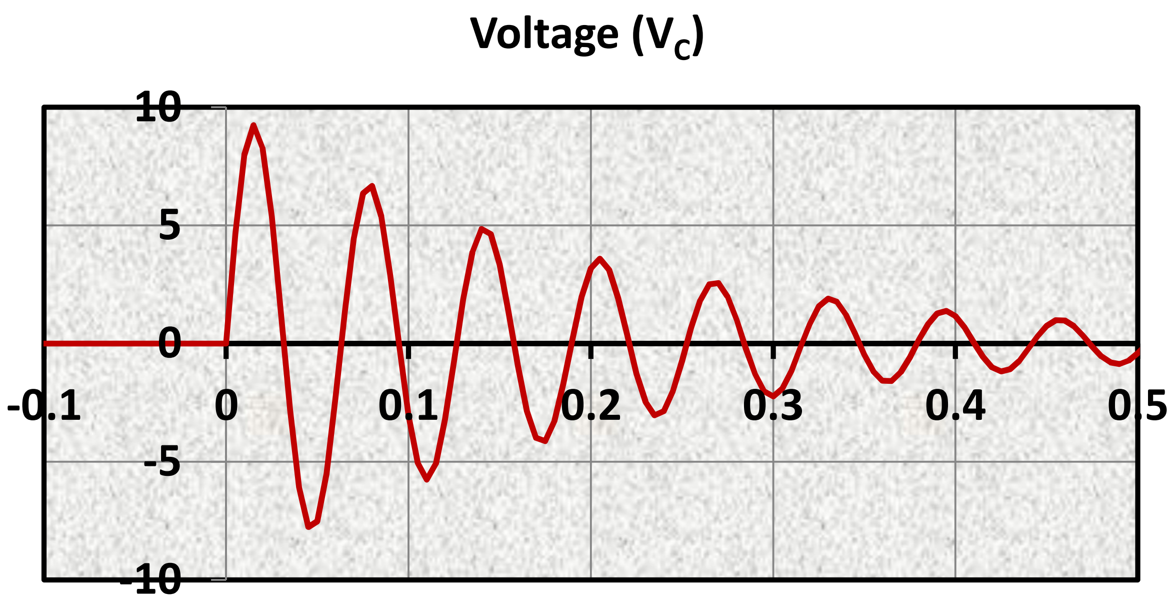
$$= 50(e^{-100t} - e^{-400t})$$

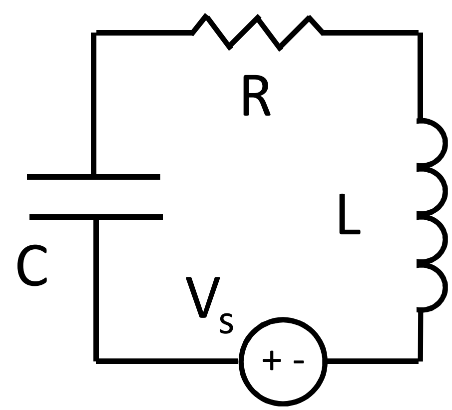
Solution



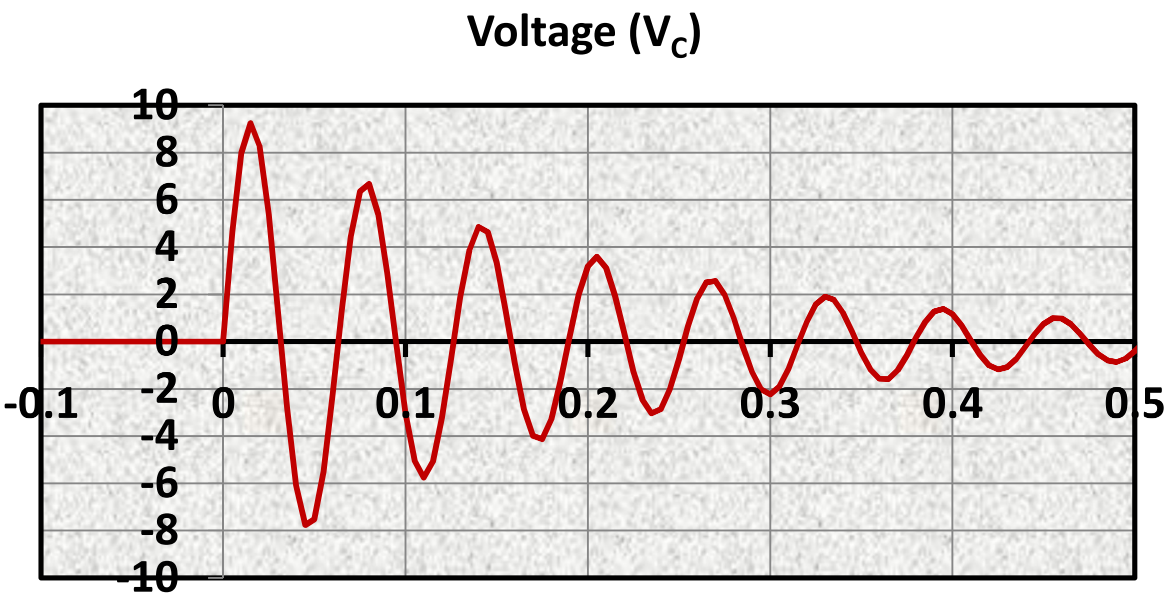
Example 2

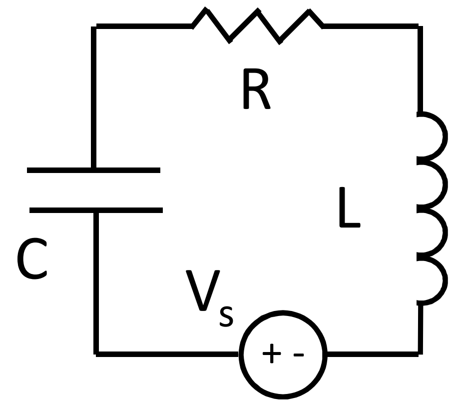
- Given the RLC voltage data below, a 1 Ω resistor, and an input of 10 V turned on at t=0, what are the approximate values of L and C?





Solution



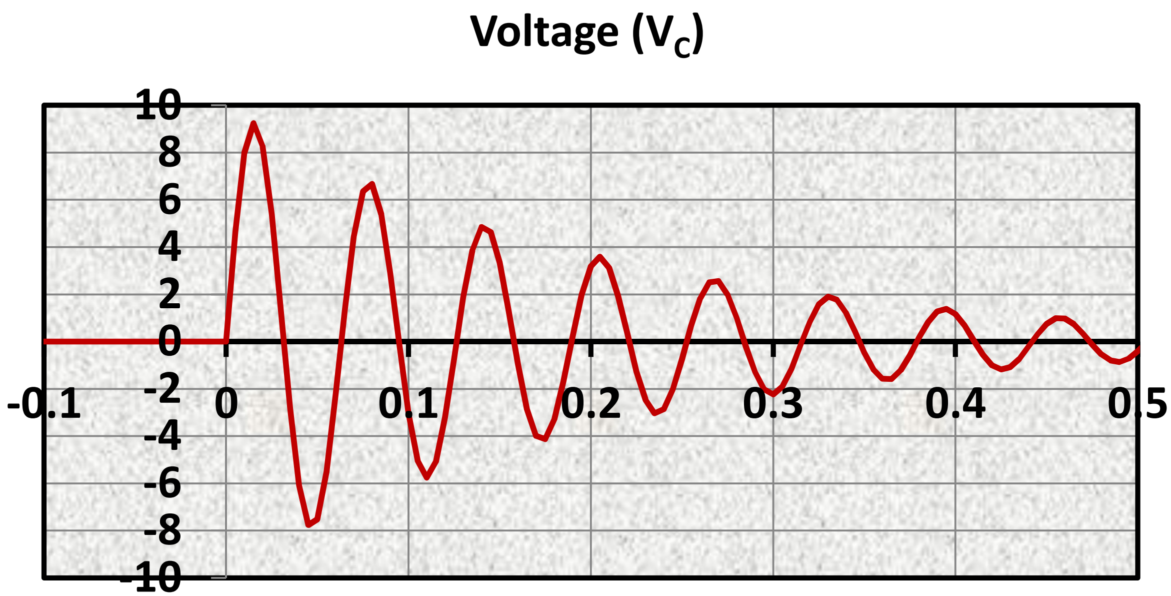


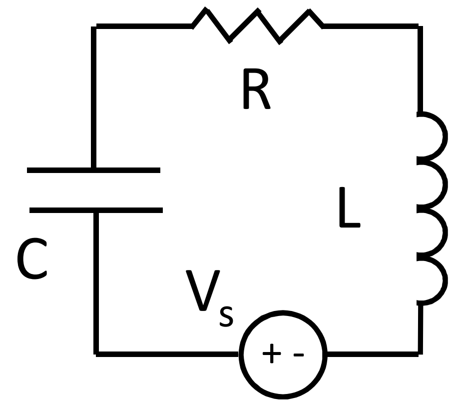
- Recall that underdamped RLC circuits obey:

$$V_C(t) = e^{-\Gamma t}\cos(\omega' t + \phi) + V_s$$

From inspection, completes 8 periods in about 0.5 seconds, implying $\omega' = 2\pi/(0.5/8) = 32\pi \approx 100$

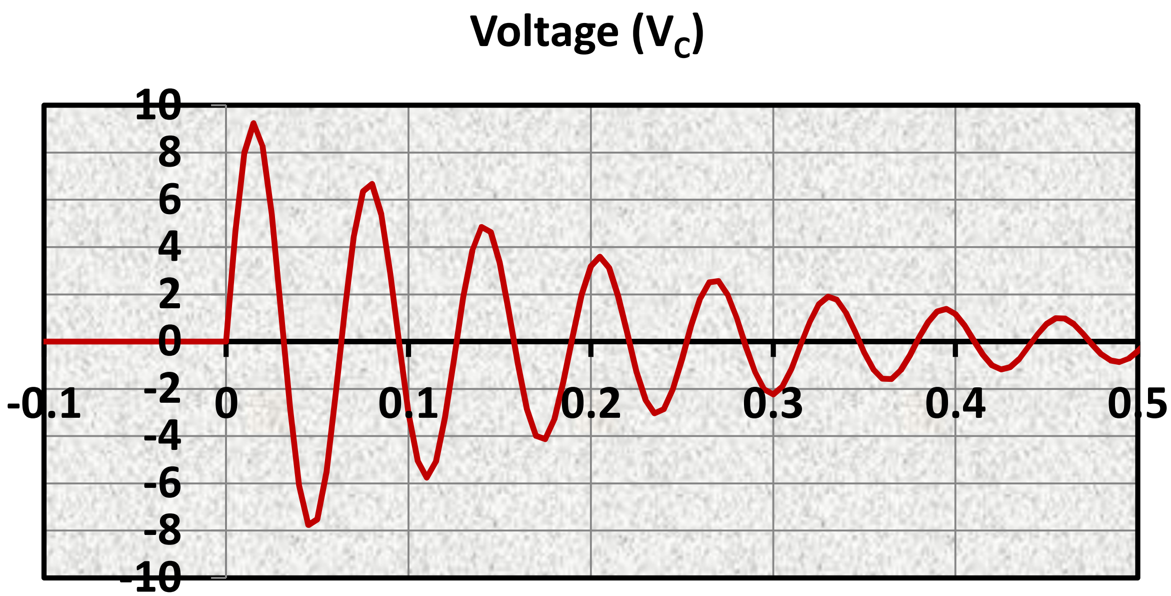
Solution

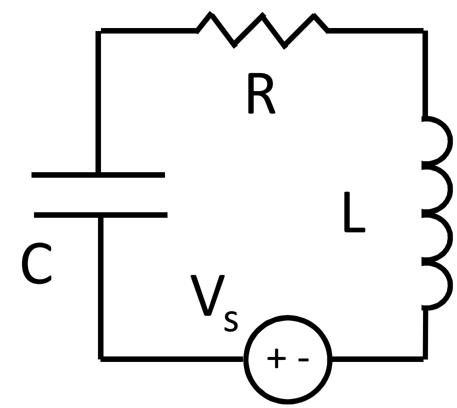




- Also decays by a factor of 10 in about 0.5 seconds. Since $\Gamma t = \ln[V_C(0)/V_C(t)] = \ln 10 \approx 2.3 = \Gamma(0.5)$, we estimate $\Gamma = 4.6$.
- Given that R=1 Ω, and $\Gamma = R/2L$, we obtain $L = \frac{R}{2\Gamma} \approx \frac{1}{2 \cdot 4.6} \approx 0.109$

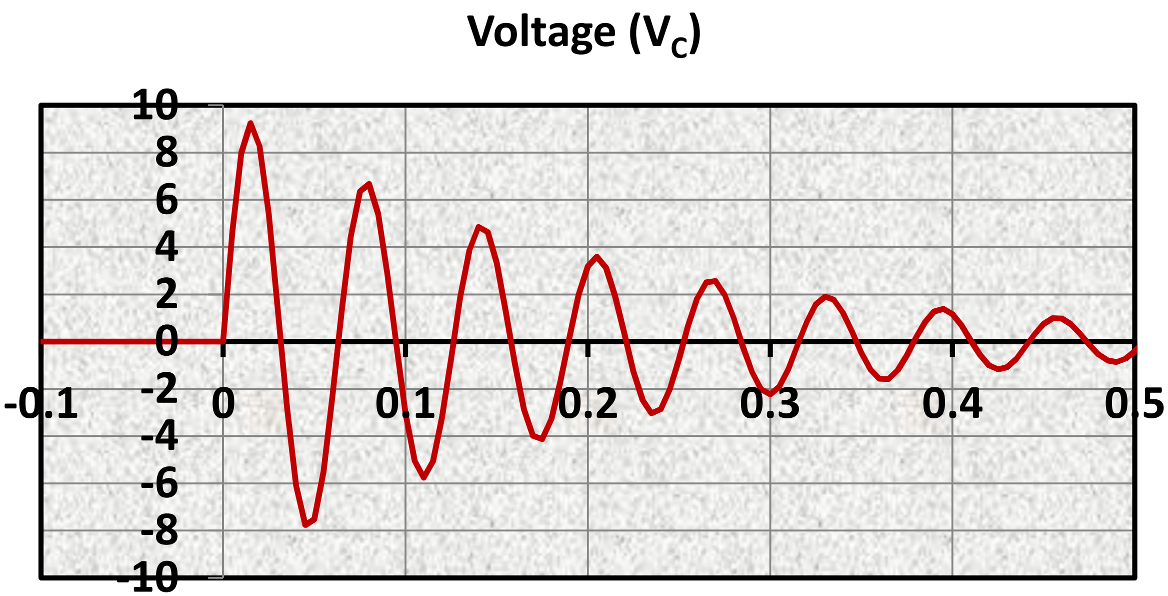
Solution

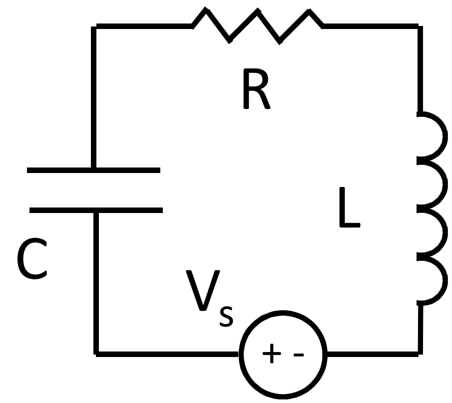




- Combining our previous results of $L \approx 0.109$ and $\omega' \approx 100$, and noting $Q \gg 1$, we can estimate $\omega' \approx \omega_o$. Since $\omega_o = \frac{1}{\sqrt{LC}}$, $C = 1/(L\omega_o^2) \approx 1/(0.109 \cdot 100^2) \approx 0.92$ mF

Solution

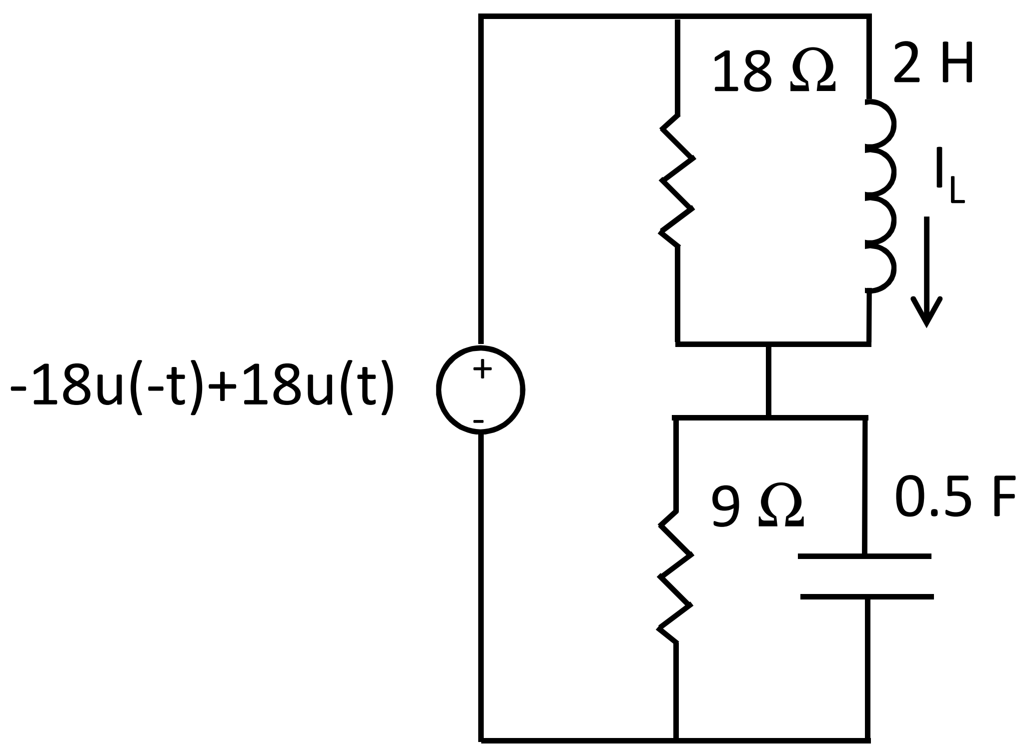




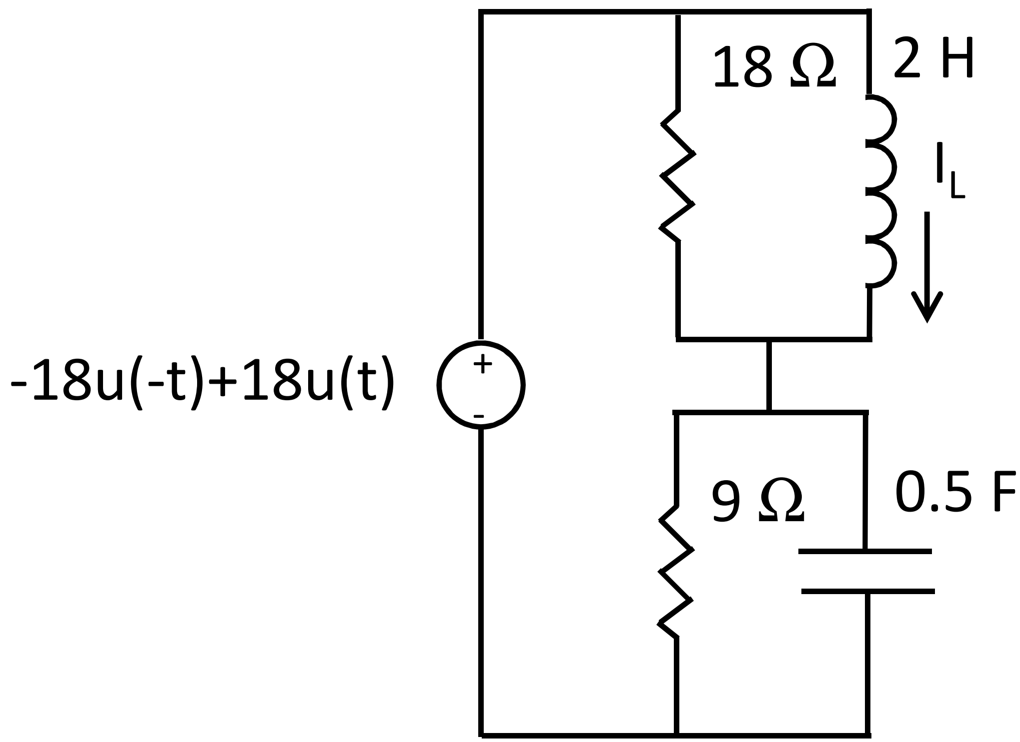
Quantity	Actual Value	Estimated value	Error
Inductance	0.1 H	0.109 H	9%
Capacitance	1 mF	0.92 mF	8%

Example 3

- For this circuit, with an independent source stepped up at t=0, what is $I_L(t)$ at all times?

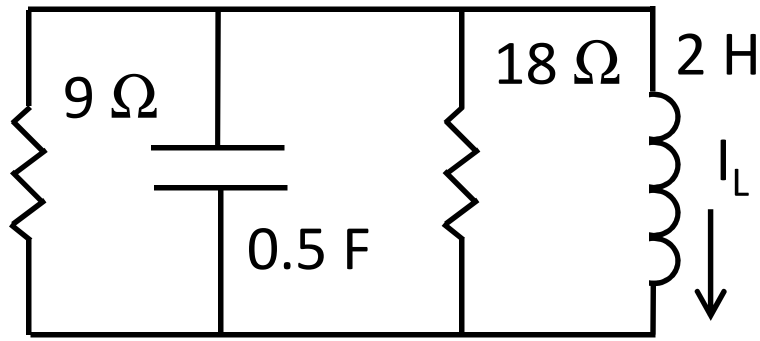


Solution



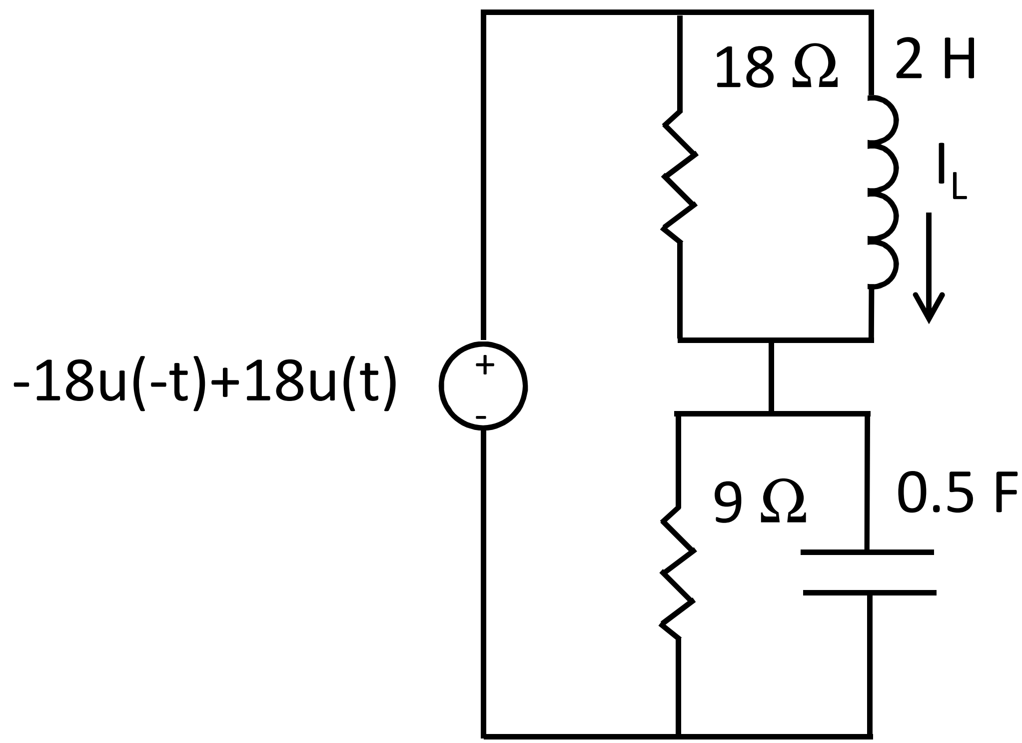
- For t<0, steady state means inductor acts as wire, $V_C(t<0)=-24$ V; $I_L(t<0)=-18/9=-2$ A
- At t=0, inductor acts as current source; capacitor as voltage source

Solution



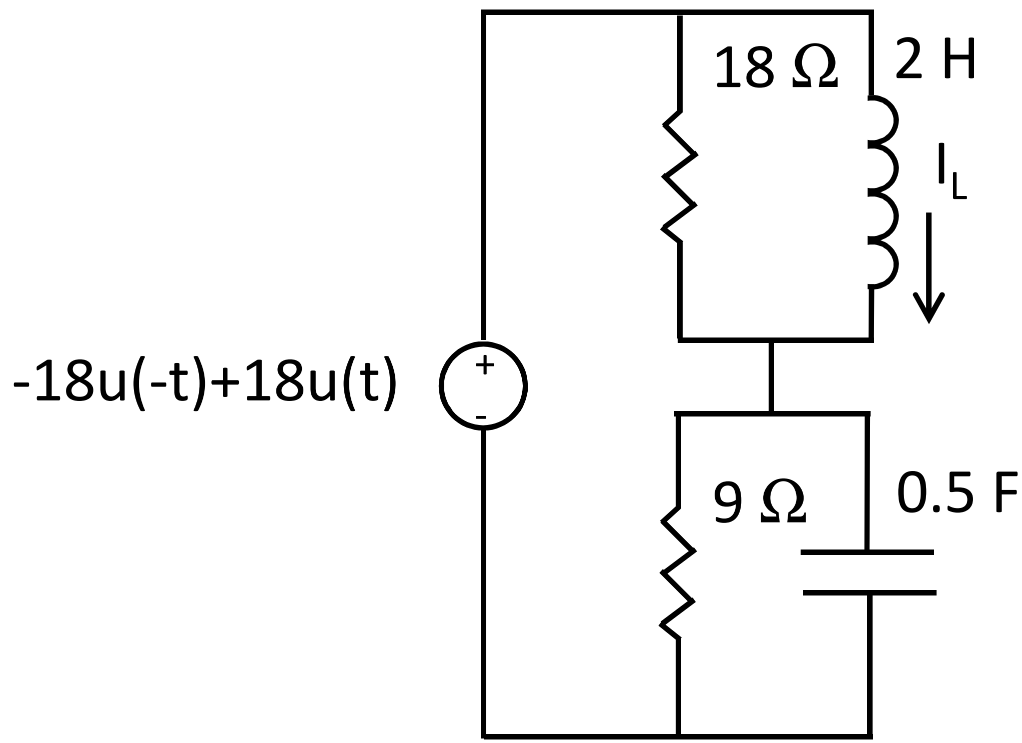
- Ignoring voltage source, circuit reduces to parallel RLC circuit, so that $\Gamma = \frac{1}{2RC} = \frac{1}{2 \cdot 6\Omega \cdot 0.5\ F} = 1/6$;
  $\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2\ H \cdot 0.5\ F}} = 1$ rad/s;
  $\omega' = \sqrt{\omega_o^2 - \Gamma^2} = \sqrt{1 - (1/6)^2} = \sqrt{35}/6$ rad/s

Solution



- We know $I_L(t) = I_o e^{-\Gamma t}\cos(\omega' t + \phi) + I_s$
- $I_L(0)$=-2 A; L dI_L/dt = [18-(-18)] V; I_s=2 A

Solution

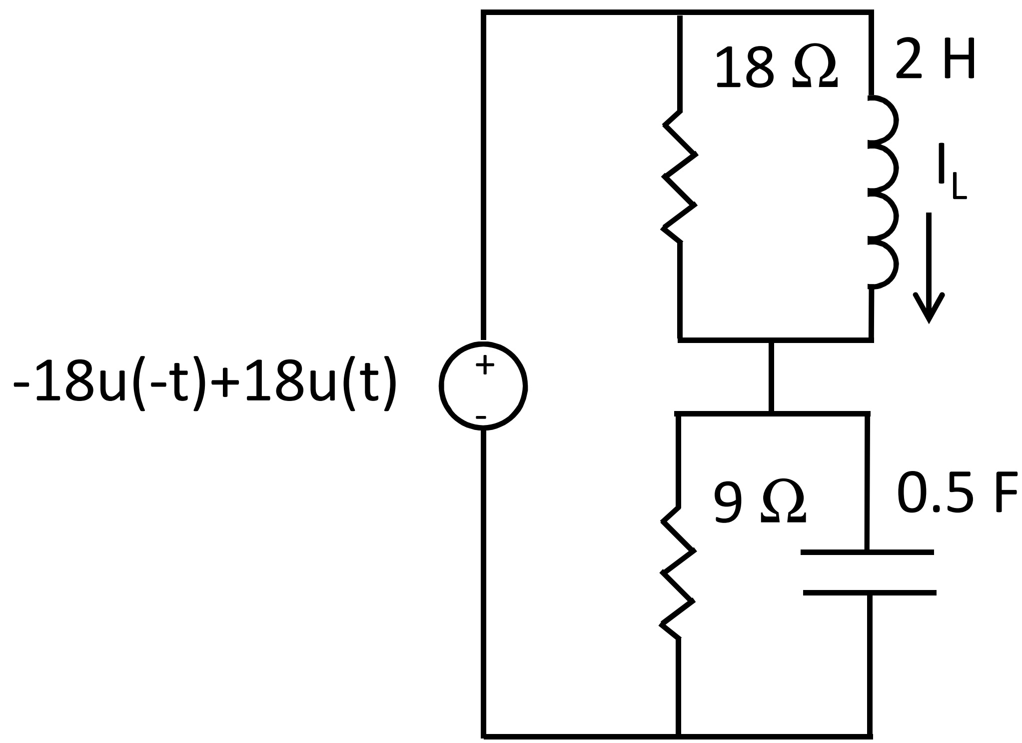


- From our initial conditions:

$$I_L(0) = -2 = 2 + I_o e^{-t/6} \cos\left(\frac{\sqrt{35}}{6} t + \phi\right)$$

$$\frac{dI_L}{dt}(0) = 18 = I_o e^{-t/6}\left[-\frac{1}{6}\cos\left(\frac{\sqrt{35}}{6} t + \phi\right) - \frac{\sqrt{35}}{6}\sin\left(\frac{\sqrt{35}}{6} t + \phi\right)\right]$$

Solution



- Set t=0 explicitly, and define $C = \cos(\phi)$:

$$-4 = I_o C$$

$$(-6) \cdot 18 = I_o\left[C - \sqrt{35(1 - C^2)}\right]$$

Solution

- Substituting yields:

$$-108 = \left(-\frac{4}{C}\right)\left[C - \sqrt{35(1 - C^2)}\right]$$

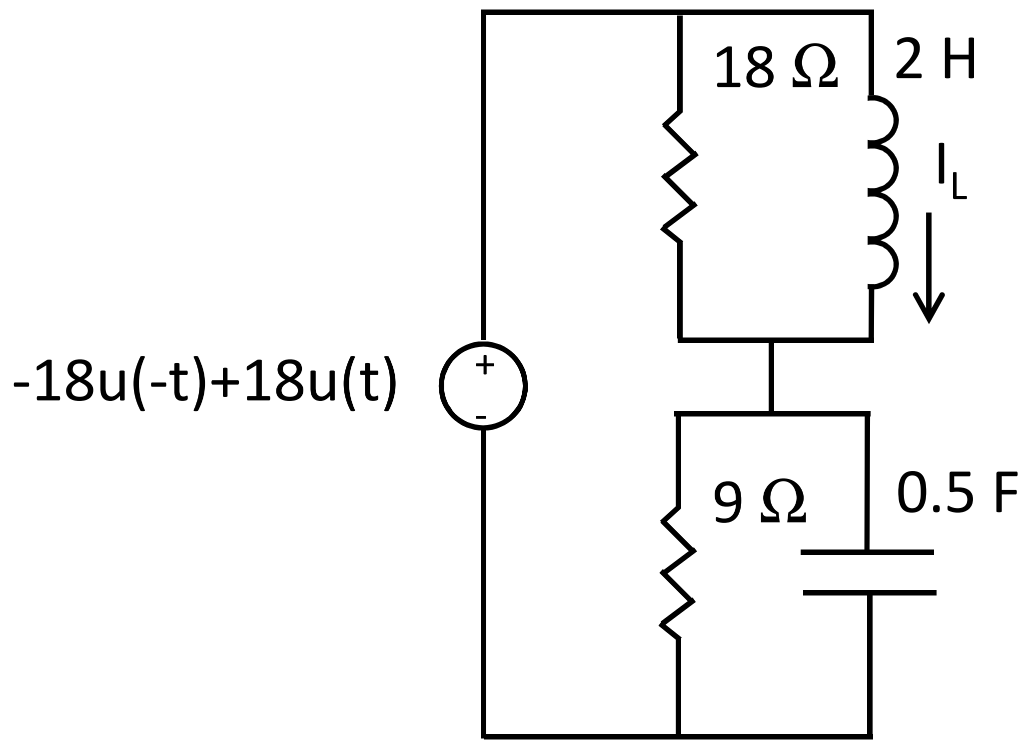
$$-104 = 4\sqrt{35\left(\frac{1}{C^2} - 1\right)}$$

$$26^2 = \frac{35}{C^2} - 35$$

$$C^2 = \frac{35}{676+35} = 0.049; C = 0.222; \phi = 77.2°$$

$$I_o = -\frac{4}{C} = -18.02$$

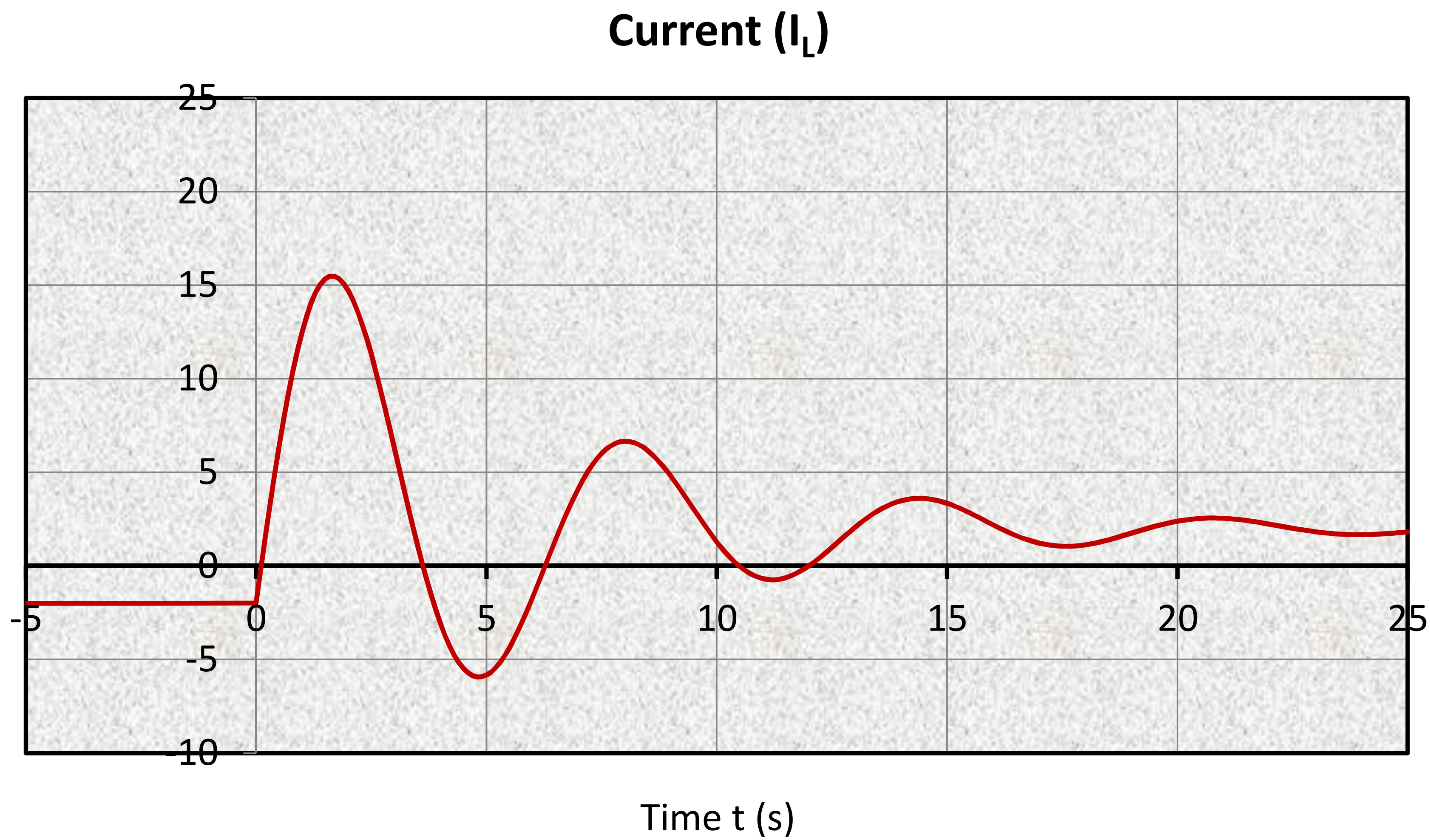
Solution



- Substituting, we obtain:

$$I_L(t) = 2 - 18.02e^{-t/6}\cos\left(\frac{\sqrt{35}}{6}t + 77.2°\right)$$

Solution



Homework

- HW #23 due today by 4:30 pm in EE 326B
- HW #24 due Wed.: DeCarlo & Lin, Chapter 9:
  - Problem 24
  - Problem 29