Driven Parallel RLC Circuits

From KCL:

\[
\begin{align*}
\frac{d^2 I_L}{dt^2} + \frac{1}{RC} \frac{dI_L}{dt} + \frac{1}{LC} I_L &= \frac{I_S}{LC} \\
\frac{d^2 I_L}{dt^2} + 2\Gamma \frac{dI_L}{dt} + \omega_0^2 I_L &= \omega_0^2 I_S \\
\Gamma &= \frac{1}{2RC}; \quad \omega_0 = \frac{1}{\sqrt{LC}}; \quad \omega' = \sqrt{\omega_0^2 - \Gamma^2} = -i\Gamma'
\end{align*}
\]

<table>
<thead>
<tr>
<th>Regime</th>
<th>Range</th>
<th>Solution</th>
<th>Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under-damped</td>
<td>$\Gamma &lt; \omega_0$</td>
<td>$I_L(t) = I_o e^{-\Gamma t} \cos(\omega' t + \phi) + I_s$</td>
<td>Oscillate &amp; decay</td>
</tr>
<tr>
<td>Critically damped</td>
<td>$\Gamma = \omega_0$</td>
<td>$I_L(t) = e^{-\Gamma t} (A_1 + A_2 t) + I_s$</td>
<td>Decay</td>
</tr>
<tr>
<td>Over-damped</td>
<td>$\Gamma &gt; \omega_0$</td>
<td>$I_L(t) = e^{-\Gamma t} (A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t}) + I_s$</td>
<td>Decay</td>
</tr>
</tbody>
</table>
Example (continued from last time)

- Consider a circuit in which input voltage switches from -10 V to +20 V at t=0 (i.e., \( V = -10 + 30u(t) \)). What is \( V_C(t) \) at all times?
Solution

\[ I_s = -\frac{1}{2} + \frac{3}{2}u(t) \]

\[ \Gamma = \frac{1}{2RC} = \frac{1}{2 \cdot 20 \cdot 10^{-4}} = 250 \]

\[ \omega_o = \frac{1}{\sqrt{0.25 \cdot 10^{-4}}} = 200 \]
Solution

\[
\begin{align*}
\text{For } t<0, \text{ inductor looks like short and } V_C(t<0)=0 \\
\text{For } t>0, \text{ non-zero voltage temporarily possible; start by solving for } I_L \text{ in overdamped regime:}
\end{align*}
\]

\[
I_L(t) = \frac{1}{\Gamma} \left( A_+ e^{\Gamma t} + A_- e^{-\Gamma t} \right) + I_S
\]

Here, \( \Gamma = 250 \), and

\[
\Gamma' = \sqrt{\Gamma^2 - \omega_0^2} = \sqrt{250^2 - 200^2} = 150
\]
Solution

Given that:

\[ I_L(t \geq 0) = e^{-250t} (A_+ e^{150t} + A_- e^{-150t}) + 1 \]
\[ = A_+ e^{-100t} + A_- e^{-400t} + 1 \]

Since \( I_L(0) = -1/2 \), and \( V_C(0) = 0 \):

\[-\frac{1}{2} = A_+ + A_- + 1 \Rightarrow A_+ = -\frac{3}{2} - A_- \]
\[0.25(-100A_+ - 400A_-) = 0 \Rightarrow A_+ = -4A_- \]

Thus, \( A_- = 1/2 \), and \( A_+ = -2 \)
Solution

\[ I_L(t \geq 0) = -2e^{-100t} + \frac{1}{2} e^{-400t} + 1 \]

- Since circuit elements are in parallel:

\[ V_c(t \geq 0) = L \frac{dI_L}{dt} \]

\[ = 0.25 \left( -2(-100)e^{-100t} + \frac{1}{2} (-400)e^{-400t} \right) \]

\[ = 50(e^{-100t} - e^{-400t}) \]
Example 2

• Given the RLC voltage data below, a 1 Ω resistor, and an input of 10 V turned on at t=0, what are the approximate values of L and C?
Solution

Recall that underdamped RLC circuits obey:

\[ V_C(t) = e^{-\Gamma t} \cos(\omega' t + \phi) + V_s \]

From inspection, completes 8 periods in about 0.5 seconds, implying \( \omega' = \frac{2\pi}{(0.5/8)} = 32\pi \approx 100 \)

\[ V_C(t) = e^{-\Gamma t} \cos(\omega' t + \phi) + V_s \]
• Also decays by a factor of 10 in about 0.5 seconds. Since $\Gamma t = \ln \left[ \frac{V_c(0)}{V_c(t)} \right] = \ln 10 \approx 2.3 = \Gamma(0.5)$, we estimate $\Gamma = 4.6$.

• Given that $R=1 \, \Omega$, and $\Gamma = R/2L$, we obtain

$$L = \frac{R}{2\Gamma} \approx \frac{1}{2 \cdot 4.6} \approx 0.109$$
Solution

• Combining our previous results of $L \approx 0.109$ and $\omega' \approx 100$, and noting $Q \gg 1$, we can estimate $\omega' \approx \omega_0$. Since $\omega_0 = \frac{1}{\sqrt{LC'}}$,

$$C = \frac{1}{(L\omega_0^2)} \approx \frac{1}{(0.109 \cdot 100^2)} \approx 0.92 \text{ mF}$$
Solution

Voltage \( (V_c) \)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Actual Value</th>
<th>Estimated value</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inductance</td>
<td>0.1 H</td>
<td>0.109 H</td>
<td>9%</td>
</tr>
<tr>
<td>Capacitance</td>
<td>1 mF</td>
<td>0.92 mF</td>
<td>8%</td>
</tr>
</tbody>
</table>
Example 3

• For this circuit, with an independent source stepped up at $t=0$, what is $I_L(t)$ at all times?

\[-18u(-t)+18u(t)\]
• For $t<0$, steady state means inductor acts as wire, $V_C(t<0)=-24$ V; $I_L(t<0)=-18/9=-2$ A
• At $t=0$, inductor acts as current source; capacitor as voltage source
Solution

\[ \text{Ignoring voltage source, circuit reduces to parallel RLC circuit, so that } \Gamma = \frac{1}{2RC} = \frac{1}{2 \cdot 6 \\Omega \cdot 0.5 \ F} = 1/6; \]

\[ \omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2 \ H \cdot 0.5 \ F}} = 1 \ \text{rad/s;} \]

\[ \omega' = \sqrt{\omega_o^2 - \Gamma^2} = \sqrt{1 - (1/6)^2} = \sqrt{35}/6 \ \text{rad/s} \]
• We know $I_L(t) = I_o e^{-\Gamma t} \cos(\omega' t + \phi) + I_s$
• $I_L(0) = -2 \ \text{A}; \ \frac{dI_L}{dt} = [18 - (-18)] \ \text{V}; \ I_s = 2 \ \text{A}$
From our initial conditions:

\[ I_L(0) = -2 = 2 + I_0 e^{-t/6} \cos \left( \frac{\sqrt{35}}{6} t + \phi \right) \]

\[ \frac{dI_L}{dt}(0) = 18 = I_0 e^{-t/6} \left[ -\frac{1}{6} \cos \left( \frac{\sqrt{35}}{6} t + \phi \right) - \frac{\sqrt{35}}{6} \sin \left( \frac{\sqrt{35}}{6} t + \phi \right) \right] \]
• Set $t=0$ explicitly, and define $C = \cos(\phi)$:

$$-4 = I_o C$$

$$(-6) \cdot 18 = I_o \left[ C - \sqrt{35(1 - C^2)} \right]$$
Solution

• Substituting yields:

\[-108 = \left(-\frac{4}{C}\right) \left[C - \sqrt{35(1-C^2)}\right]\]

\[-104 = 4 \sqrt{35 \left(\frac{1}{C^2} - 1\right)}\]

\[26^2 = \frac{35}{C^2} - 35\]

\[C^2 = \frac{35}{676+35} = 0.049; \quad C = 0.222; \quad \phi = 77.2^\circ\]

\[I_o = -\frac{4}{C} = -18.02\]
Solution

-18u(-t) + 18u(t)

\[ I_L(t) = 2 - 18.02e^{-t/6} \cos \left( \frac{\sqrt{35}}{6} t + 77.2^\circ \right) \]
Solution

Current ($I_L$)

Time $t$ (s)
Homework

• HW #23 due today by 4:30 pm in EE 326B
• HW #24 due Wed.: DeCarlo & Lin, Chapter 9:
  – Problem 24
  – Problem 29