Who Uses Complex Numbers?

• Cardano and Bombelli discovered ‘fictitious’ numbers solving cubic equations in 1500’s
• Euler, Wessel, Argand and Gauss spread modern understanding in 1700’s
• Found widespread use in physics and engineering in 20th century because of:
  – Electromagnetism
  – Sound Waves
  – Quantum Mechanics
  – Signal processing
Why Use Complex Numbers?

• Simplifies mathematical representation of sinusoidal oscillation
• Allows for solution of range of difficult problems with powerful theorems:
  – MacLaurin and Taylor theorems
  – Residue theorem
When and Where Should You Use Complex Numbers

- In solving trigonometric problems
- In solving second order differential equations
- When problems are otherwise intractable
- Can be used in any problem involving reals – when in doubt, try it out
What Are Complex Numbers?

• Pairs of real numbers in complex plane:
  – Cartesian representation: $z=x+iy$
  – Polar representation: $z=re^{i\theta}$

• Converting between representations:
  – Polar to Cartesian: $x=r \cos(\theta)$; $y=r \sin(\theta)$
  – Cartesian to Polar: $r=(x^2+y^2)^{1/2}$; $\theta=\tan^{-1}(y/x)$
How Do We Use Complex Numbers?

• Complex extension of unary (1 term) arithmetic:
  – Exponentiation: $z^n = (re^{i\theta})^n = r^n e^{in\theta}$
  – Absolute value: $|z| = |re^{i\theta}| = r = |x+iy| = (x^2+y^2)^{1/2}$
  – Natural logarithm: $\ln(z) = \ln(re^{i\theta}) = \ln(r) + i\theta$

• Complex extension of binary (2 term) arithmetic:
  – Addition: $z_1 + z_2 = (x_1+iy_1) + (x_2+iy_2) = x_1 + x_2 + i(y_1 + y_2)$
  – Subtraction: $z_1 + z_2 = (x_1+iy_1) + (x_2+iy_2) = x_1 + x_2 + i(y_1 + y_2)$
  – Multiplication: $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
  – Division: $z_1 / z_2 = r_1 e^{i\theta_1} / r_2 e^{i\theta_2} = (r_1 / r_2) e^{i(\theta_1 - \theta_2)}$

• Additional operations:
  – Complex conjugation: $z^* = (x+iy)^* = x-iy$
Complex Exponentials

• Some key equations:

\[ e^{i \omega t} = \cos \omega t + i \sin \omega t \]

\[ e^{(i \omega - \Gamma)t} = e^{-\Gamma t} (\cos \omega t + i \sin \omega t) \]

\[ \cos \omega t = \frac{e^{i \omega t} + e^{-i \omega t}}{2} \]

\[ \sin \omega t = \frac{e^{i \omega t} - e^{-i \omega t}}{2i} \]
Deriving Trigonometric Identities

Can use Euler’s equation to write:

\[ e^{i(\alpha + \beta)} = e^{i\alpha} e^{i\beta} \]

\[ = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \]

\[ = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]

\[ + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta) \]

Which yields:

\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]

\[ \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \]
Example #1

• How does this series RC circuit respond to an AC current source turned on at t=0?

\[ 1 \text{ mF} \]

\[ 1 \text{ k}\Omega \]

\[ 3 \ u(t) \cos(t) \]
Solution

• From KCL, we obtain:
\[ RC \frac{dV}{dt} + V = 3u(t) \cos t \]

• Recall the homogeneous solution is:
\[ V(t) = V_o e^{-t/\tau}, \text{ where } \tau = RC = 1 \text{ s} \]

• We can guess the driven solution is:
\[ V(t) = A(t) \cos t = \text{Re}[A(t)e^{it}] \]
Solution

• Substituting $V(t) = A(t)e^{it}$:
  \[
  \left( \frac{dA}{dt} + iA + A \right) e^{it} = 3u(t)e^{it}
  \]
  \[
  A = \frac{3}{1 + i}
  \]

• Combining our solutions yields:
  \[
  V(t) = \frac{3}{1 + i} e^{it} + V_o e^{-t}
  \]

• Applying our initial condition ($V(0)=0$) yields:
  \[
  V_o = -\frac{3}{1 + i}
  \]
Solution

• Thus, we obtain:

\[ V(t) = \begin{cases} 
\frac{3}{1 + i} \left[ e^{it} - e^{-t} \right], & t > 0 \\
0, & \text{otherwise}
\end{cases} \]

• The physical value of \( V(t) \) is the real part of the complex expression
Solution

Voltage (V)

Time t (s)
Example #2

• How does this parallel RLC circuit respond to an AC current source?

\[ I_L = 2\cos(120\pi t) \]
Solution

From KCL:

\[
\frac{d^2 I_L}{dt^2} + 2\Gamma \frac{dI_L}{dt} + \omega_o^2 I_L = \omega_o^2 I_S
\]

\[
\Gamma = \frac{1}{2RC} = \frac{1}{2 \cdot 1 \text{k}\Omega \cdot 1 \text{mF}} = 0.5 \text{ s}
\]

\[
\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20 \text{ mH} \cdot 1 \text{ mF}}} = 223.6 \text{ rad/s}
\]

\[
\omega' = \sqrt{\omega_o^2 - \Gamma^2} = \sqrt{223.6^2 - 0.5^2} = 223.6 \text{ rad/s}
\]
Solution

Postulating a solution of the form:

\[ I_L = I_o e^{120\pi it} \]

We substitute into:

\[ \frac{d^2 I_L}{dt^2} + \frac{dI_L}{dt} + 223.6^2 I_L = 223.6^2 I_s \]

To obtain:

\[ [(120\pi i)^2 + 120\pi i + 223.6^2]I_o e^{120\pi it} = 223.6^2 \cdot 2e^{120\pi it} \]

\[ I_o = \frac{2 \cdot 223.6^2}{223.6^2 - (120\pi)^2 + 120\pi i} \approx -1.085 \]
Phasors

• Shorthand for writing complex numbers
• Closely connected with polar representation
• General form:
  \[ \mathbf{V} = V_m \angle \phi = V_m e^{j(\omega t + \phi)} \]
• Example: for a voltage source given by \( 5 \cos(\omega t + 30^\circ) \), the phasor \( \mathbf{V} = 5 \angle 30^\circ \)
KCL for Phasors

• Generalization of KCL from before
• At each current junction or Gaussian surface:

\[ \sum_{k=1}^{N} I_k(t) = \sum_{k=1}^{N} I_{km} \angle \phi_k = 0 \]
Phasor KCL Example

• Consider 2 input currents with phasors given by $6\angle15^\circ$ A and $-10\angle45^\circ$ A. What is the total output current?
Phasor KCL Solution

- Assuming $\omega t=0$, we can convert to Cartesian coordinates with $x = r \cos \phi$ and $y = r \sin \phi$:
  
  \[
  6\angle 15^\circ = 6 \cos 15^\circ + j \cdot 6 \sin 15^\circ = 5.80 + 1.55j \\
  -10\angle 45^\circ = -10 \cos 45^\circ + j \cdot (-10) \sin 45^\circ = -7.07 - 7.07j \\
  6\angle 15^\circ + -10\angle 45^\circ = (5.80 - 7.07) + (1.55 - 7.07)j \\
  6\angle 15^\circ + -10\angle 45^\circ = -1.27 - 5.52j
  \]

- We can convert back with $r = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1} y/x$:
  
  \[
  6\angle 15^\circ + -10\angle 45^\circ = 5.66\angle -103^\circ = -5.66\angle 77^\circ
  \]
Homework

• HW #28 due today by 4:30 pm in EE 325B
• HW #29 due Mon.:
  – DeCarlo and Lin, Chapter 8, Problem 38
  – The op amp in the following circuit is ideal. The voltage $v_{IN}(t)$ has been toggling between -0.2V and 0.2V for a long time (as shown in the plot). Find and plot $v_{out}(t)$. Label the axes.