ECE 201, Section 3
Lecture 30

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Recap From Last Time

- Solving first AC circuits – solution comprised of two parts:
  – Transient (exponentially decaying) solution
  – Sinusoidal steady state solution, tracking source frequency
- Best to treat everything in terms of complex exponentials; can be represented and manipulated in Cartesian and polar coordinates

Phasors

- Shorthand for writing complex numbers
- Closely connected with polar representation
- General form:

$$\boldsymbol{V} = V_m \angle \phi = V_m e^{j(\omega t + \phi)}$$

- Example: for a voltage source given by $5\cos(\omega t + 30°)$, the phasor $\boldsymbol{V} = 5\angle 30°$

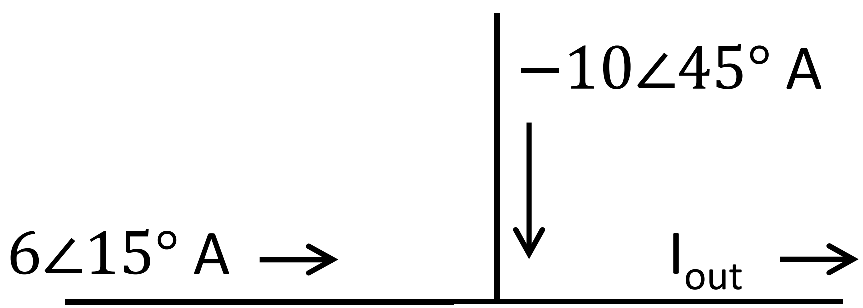
KCL for Phasors

- Generalization of KCL from before
- At each current junction or Gaussian surface:

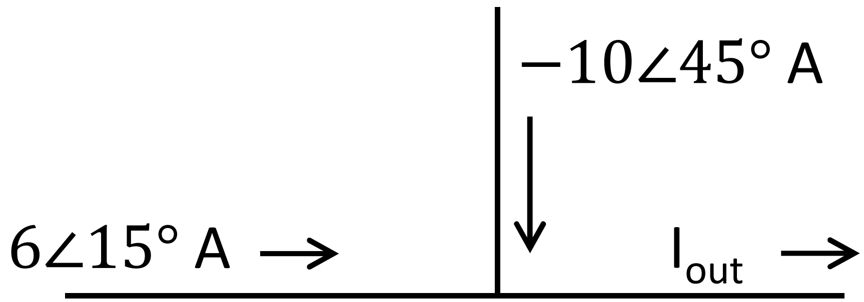
$$\sum_{k=1}^{N} \boldsymbol{I}_k(t) = \sum_{k=1}^{N} I_{km}\angle\phi_k = 0$$

Phasor KCL Example

- Consider 2 input currents with phasors given by $6\angle 15°$ A and $-10\angle 45°$ A. What is the total output current?



Phasor KCL Solution



- Assuming ωt=0, we can convert to Cartesian coordinates with $x = r\cos\phi$ and $y = r\sin\phi$:

$$6\angle 15° = 6\cos 15° + j \cdot 6\sin 15° = 5.80 + 1.55j$$

$$-10\angle 45° = -10\cos 45° + j \cdot (-10)\sin 45° = -7.07 - 7.07j$$

$$6\angle 15° + -10\angle 45° = (5.80 - 7.07) + (1.55 - 7.07)j$$

$$6\angle 15° + -10\angle 45° = -1.27 - 5.52j$$

- We can convert back with $r = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1} y/x$:

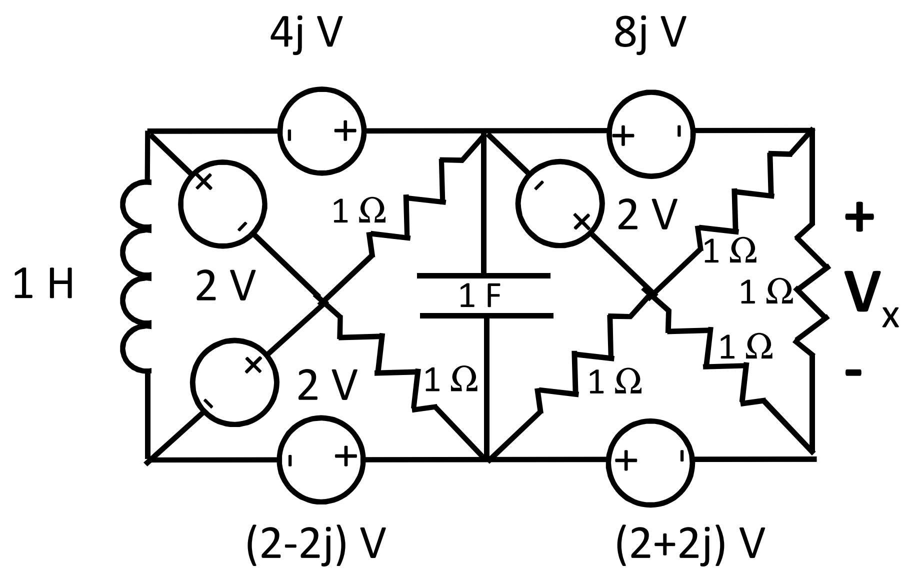
$$6\angle 15° + -10\angle 45° = 5.66\angle -103° = -5.66\angle 77°$$

KVL for Phasors

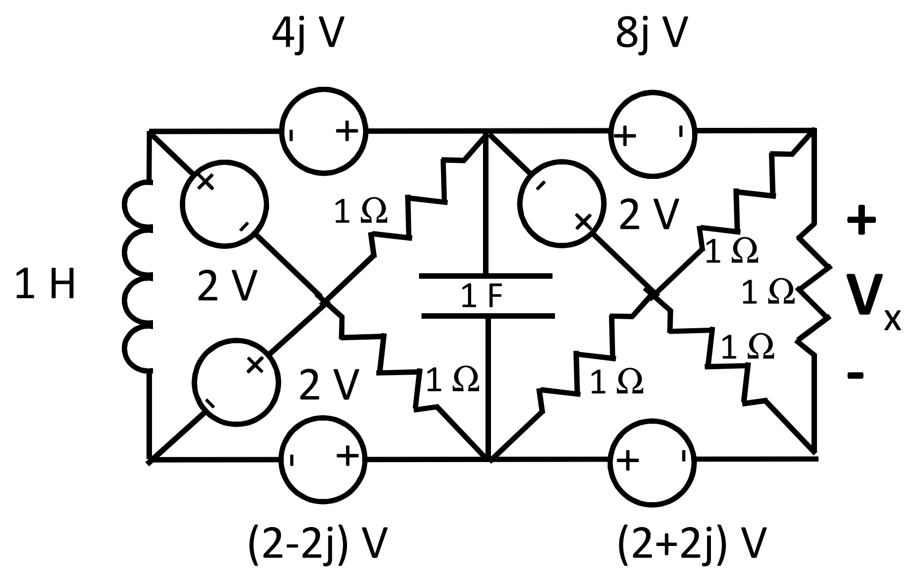
- Generalization of KVL from before to phasors:
  - Voltage difference phasor $\boldsymbol{V}_{AB}$ between points A and B is the same, regardless of the path taken
  - The voltage difference is directionally sensitive, i.e., $\boldsymbol{V}_{AB} = -\boldsymbol{V}_{BA}$
  - The voltage difference over a closed loop is zero (otherwise, voltage would be non-unique)

Phasor KVL Example

- What is the phasor voltage $\boldsymbol{V}_x$ in this circuit?



Phasor KVL Solution



- Assign - terminal of $\boldsymbol{V}_x$ to ground
- Find path leading to + terminal of $\boldsymbol{V}_x$
- Keeping track of signs:

$$\boldsymbol{V}_x = (2 + 2j) - (2 - 2j) + 2 + 2 + 4j - 8j = 4\ \mathrm{V}$$

Phasor Impedance

- Basic concept: generalize resistance to the concept of impedance for phasors, such that:

$$\boldsymbol{V} = Z(j\omega)\boldsymbol{I}$$

- For resistor, no conservation rules mean no lag and simple relation between impedance and resistance:

$$Z(j\omega) = R, \qquad \text{for all } \omega$$

Phasor Impedance: Inductors

- Given that we have impedance given by:

$$\boldsymbol{V} = Z(j\omega)\boldsymbol{I}$$

- For inductors, we have:

$$V = L\frac{dI}{dt}$$

- Assuming an input current source $I = Ae^{j(\omega t+\phi)}$:

$$V = (Lj\omega)Ae^{j(\omega t+\phi)} = j\omega LI$$

- Comparing the first & third expressions yields:

$$Z(j\omega) = j\omega L, \qquad \text{for all } \omega$$

- Thus, voltage leads current by $90°$

Phasor Impedance: Capacitors

- Given that we have impedance given by:

$$\boldsymbol{V} = Z(j\omega)\boldsymbol{I}$$

- For capacitors, we have:

$$I = C\frac{dV}{dt}$$

- Assuming an input voltage source $V = Ae^{j(\omega t+\phi)}$:

$$I = (Cj\omega)Ae^{j(\omega t+\phi)} = j\omega CV$$

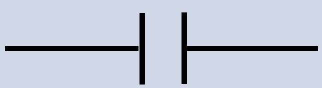
- Comparing the first & third expressions yields:

$$Z(j\omega) = \frac{1}{j\omega C}, \qquad \text{for all } \omega$$

- Thus, current leads voltage by 90°

Phasor Admittance

- Inverse of impedance
- Symbolized by $Y(j\omega) = 1/Z(j\omega)$

Circuit Element	Impedance	Admittance
	$Z(j\omega) = R$	$Y(j\omega) = \dfrac{1}{R}$
	$Z(j\omega) = j\omega L$	$Y(j\omega) = \dfrac{1}{j\omega L}$
	$Z(j\omega) = \dfrac{1}{j\omega C}$	$Y(j\omega) = j\omega C$

Impedance Properties

- For general linear networks:
  - Resistance $R = \operatorname{Re} Z(j\omega)$ (from resistors)
  - Reactance $X = \operatorname{Im} Z(j\omega)$ (from inductors, capacitors)
- Impedances add in series:

$$Z_{eq}(j\omega) = \sum_{k=1}^{N} Z_k(j\omega)$$

- Admittances add in parallel:

$$Y_{eq}(j\omega) = \sum_{k=1}^{N} Y_k(j\omega)$$

Impedance Properties

- For circuit elements in series, voltage division rule becomes:

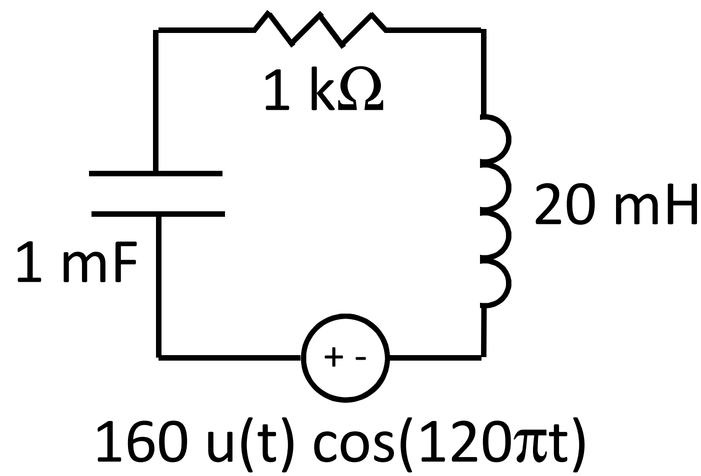
$$V_k = \frac{Z_k}{Z_{eq}} V_{tot}$$

- For circuit elements in parallel, current division rule becomes:

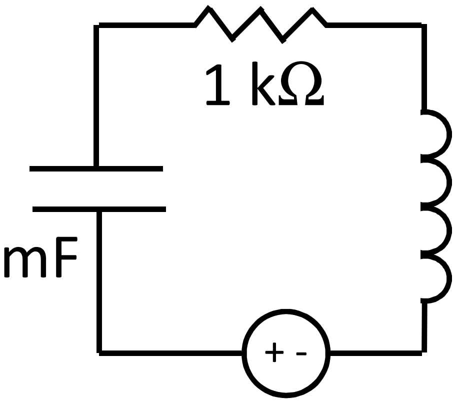
$$I_k = \frac{Y_k}{Y_{eq}} I_{tot}$$

Impedance Example

- What is the impedance across the source terminals of this circuit? After the source is turned on, what is its steady-state response?



Impedance Solution



- Since impedances add in series, we get:

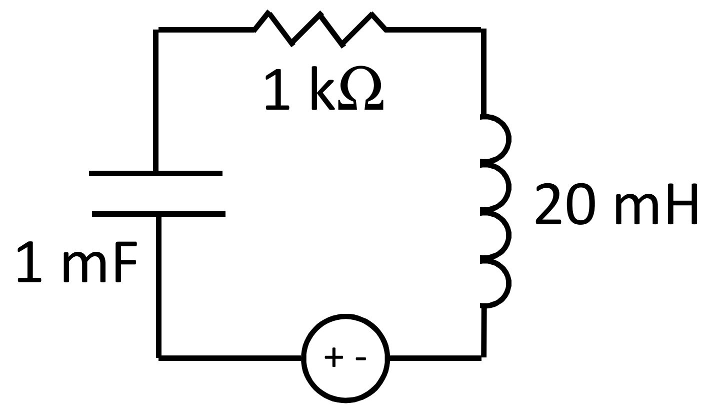
$$Z_{eq} = 1\ \text{k}\Omega + j\omega(20\ \text{mH}) + 1/[j\omega(1\ \text{mF})]$$

- If $\omega = 120\pi$, we get:

$$Z_{eq} = 1\ \text{k}\Omega + j120\pi(20\ \text{mH}) + 1/[j120\pi(1\ \text{mF})]$$

$$Z_{eq} = 10^3 + 4.887j\ \Omega$$

Impedance Solution



- Given that:

$$Z_{eq} = 10^3 + 4.887j\ \Omega$$

- Applying the definition of impedance:

$$I = \frac{V}{Z_{eq}} = \frac{160e^{j120\pi t}}{10^3 + 4.887j}$$

$$I = 0.160e^{j(120\pi t - 0.28°)} = 0.16\angle -0.28°$$

Homework

- HW #29 due today by 4:30 pm in EE 325B
- HW #30 due Wed:

1. Add the complex numbers $z_1 = 8e^{-j\pi/6}$ and $z_2 = -5e^{j5\pi/8}$. Express your answer (a) in real and imaginary parts, and (b) as a magnitude and phase. Draw z_1, z_2 and $z_1 + z_2$ on the complex plane. Label the real part, the imaginary part, and magnitude and the phase of $z_1 + z_2$.
2. Multiply the complex numbers $z_1 = 2-4j$ and $z_2 = -4+3j$. Express your answer (a) in real and imaginary parts, and (b) as a magnitude and phase. (c) Compare $|z_1||z_2|$ to $|z_1 z_2|$. (d) What is the relationship between the phase of z_1, the phase of z_2, and the phase of $z_1 z_2$?
3. For z_1 and z_2 given in problem 2, determine z_1/z_2. Express your answer (a) in real and imaginary parts, and (b) as a magnitude and phase. (c) Compare $|z_1|$ and $|z_2|$ to $|z_1/z_2|$. (d) What is the relationship between the phase of z_1, the phase of z_2, and the phase of z_1/z_2?