Exam #3

• Thurs., Nov. 15 at 6:30 pm in LILY 1105
• No class or HW next week (Nov. 19-23)!
• Covers Lectures 22-32, with 3 major topics:
  – Second-order circuits (primarily RLC)
  – Op-amps
  – Sinusoidal steady state
• Posted 7 sample exams with answers for Spring 2009, 2010, and 2011, and full solutions for ‘09 and ‘11
General Solutions: RC & RL Circuits

\[ X(t) = X_\infty + [X(t_0^+) - X_\infty]e^{-(t-t_0)/\tau} \]

Where \( X(t) \) is current or voltage at time \( t \)

- Voltages are continuous across capacitors; currents are continuous through inductors
- Capacitor currents and inductor voltages can jump discontinuously
Driven Series RLC Circuits

From KVL:

\[
\frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{1}{LC} Q = \frac{V_s}{L} \\
\frac{d^2V_c}{dt^2} + 2\Gamma \frac{dV_c}{dt} + \omega_o^2 V_c = \omega_o^2 V_s
\]

\[\Gamma = \frac{R}{2L}; \ \omega_o = \frac{1}{\sqrt{LC}}; \ \omega' = \sqrt{\frac{\omega_o^2}{\omega_o^2 - \Gamma^2}} = -i\Gamma'
\]

<table>
<thead>
<tr>
<th>Regime</th>
<th>Range</th>
<th>Solution</th>
<th>Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under-damped</td>
<td>(\Gamma &lt; \omega_o)</td>
<td>(V_C(t) = V_o e^{-\Gamma t} \cos(\omega' t + \phi) + V_s)</td>
<td>Oscillate &amp; decay</td>
</tr>
<tr>
<td>Critically</td>
<td>(\Gamma = \omega_o)</td>
<td>(V_C(t) = e^{-\Gamma t}(A_1 + A_2 t) + V_s)</td>
<td>Decay</td>
</tr>
<tr>
<td>Over-damped</td>
<td>(\Gamma &gt; \omega_o)</td>
<td>(V_C(t) = e^{-\Gamma t}(A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t}) + V_s)</td>
<td>Decay</td>
</tr>
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</table>
Driven Parallel RLC Circuits

From KCL:

\[
\begin{align*}
\frac{d^2}{dt^2}I_L + \frac{1}{RC} \frac{d}{dt}I_L + \frac{1}{LC}I_L &= \frac{I_S}{LC} \\
\frac{d^2}{dt^2}I_L + 2\Gamma \frac{d}{dt}I_L + \omega_o^2I_L &= \omega_o^2I_S
\end{align*}
\]

\[\Gamma = \frac{1}{2(2RC)}; \quad \omega_o = \frac{1}{\sqrt{LC}}; \quad \omega' = \sqrt{\omega_o^2 - \Gamma^2} = -i\Gamma'\]

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<tr>
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<td>Oscillate &amp; decay</td>
</tr>
<tr>
<td>Critically damped</td>
<td>(\Gamma = \omega_o)</td>
<td>(I_L(t) = e^{-\Gamma t}(A_1 + A_2 t) + I_s)</td>
<td>Decay</td>
</tr>
<tr>
<td>Over-damped</td>
<td>(\Gamma &gt; \omega_o)</td>
<td>(I_L(t) = e^{-\Gamma t}(A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t}) + I_s)</td>
<td>Decay</td>
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Phasor Review

- Shorthand for writing complex numbers:
  \[ V = V_m \angle \phi = V_m e^{j(\omega t + \phi)} \]
- Ohm’s law with phasors:
  \[ V = Z(j\omega)I \\
  I = Y(j\omega)V \]

<table>
<thead>
<tr>
<th>Circuit Element</th>
<th>Impedance</th>
<th>Admittance</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="placeholder" alt="Resistor" /></td>
<td>( Z(j\omega) = R )</td>
<td>( Y(j\omega) = \frac{1}{R} )</td>
</tr>
<tr>
<td><img src="placeholder" alt="Inductor" /></td>
<td>( Z(j\omega) = j\omega L )</td>
<td>( Y(j\omega) = \frac{1}{j\omega L} )</td>
</tr>
<tr>
<td><img src="placeholder" alt="Capacitor" /></td>
<td>( Z(j\omega) = \frac{1}{j\omega C} )</td>
<td>( Y(j\omega) = j\omega C )</td>
</tr>
</tbody>
</table>
Impedance Properties

• For circuit elements in series, voltage division rule becomes:

\[ V_k = \frac{Z_k}{Z_{eq}} V_{tot} \]

• For circuit elements in parallel, current division rule becomes:

\[ I_k = \frac{Y_k}{Y_{eq}} I_{tot} \]
Second Order Circuit Overview

\[ V_i \cos \omega t \]

usual circuit model

set up DE

\[ |V_p| \cos[\omega t + \angle V_p] \]

nightmare trig.

ej\omega t drive

\[ V_i e^{j\omega t} \]

complex algebra

take real part

impedance-based circuit model

complex algebra

Adapted from A. Agarwal & J. Lang, Course Materials for MIT 6.002, Spring 2007
Ideal Op-Amps

• Golden rules:
  – Both input currents are zero
  – For closed loops: both input voltages are equal
1. At $t = 0$ sec, the inductor current is $i_L(0^+) = 5A$ and the capacitor voltage is $v_c(0^+) = 0V$. Find $v_c(t)$ for $t \geq 0$ s (in V).

$$v_c(t) = 25\cos(10^6 t) + 25\sin(10^6 t)$$

$$v_c(t) = 25\cos(10^6 t)$$

$$v_c(t) = 25\sin(10^6 t)$$

$$v_c(t) = 12.5\cos(10^3 t) + 12.5\sin(10^3 t)$$

$$v_c(t) = 12.5\cos(10^3 t)$$

$$v_c(t) = -12.5\sin(10^3 t)$$

$$v_c(t) = 0$$
11. Calculate the equivalent impedance, $Z_{eq}$ (in $\Omega$), for the circuit below.

(1) $2 \angle 53.13^\circ$  (2) $2 \angle -53.13^\circ$  (3) $2 \angle 36.7^\circ$  (4) $2 \angle -36.7^\circ$

(5) $1 \angle 40^\circ$  (6) $1 \angle -40^\circ$  (7) $1 \angle 45^\circ$
13. The circuit shown below is in steady state. Find $v_L(t)$.

![Circuit Diagram]

(1) $v_L(t) = \sqrt{2} \cos(100t + 90^\circ)$
(2) $v_L(t) = 2 \cos(100t + 90^\circ)$
(3) $v_L(t) = \sqrt{2} \cos(100t + 135^\circ)$
(4) $v_L(t) = 2 \cos(100t + 135^\circ)$
(5) $v_L(t) = \sqrt{2} \cos(100t + 170^\circ)$
(6) $v_L(t) = 2 \cos(100t + 170^\circ)$
(7) $v_L(t) = \sqrt{2} \cos(200t + 170^\circ)$
Equation Sheet for Exam

I=dQ/dt
V=IR or V=IZ; I=GV or I=VV
Q=CV and V=L dI/dt

In series:
\[ Z_{eq} = \Sigma_k Z_k \]
\[ V_k = VZ_k / Z_{eq} \]

In parallel:
\[ Y_{eq} = \Sigma_k Y_k \]
\[ I_k = VY_k = IY_k / Y_{eq} \]
\[ P = IV \]

For first-order circuits:
\[ X = X_\infty + (X_0 - X_\infty) e^{-(t-t_0)/\tau} \]
For RL circuits: \( \tau = L/R \)
For RC circuits: \( \tau = RC \)

For second-order (RLC) circuits:
\[ X(t) = A e^{-\Gamma t} \cos(\omega' t + \phi) + X_s \]
\[ X(t) = (A_1 + A_2 t) e^{-\Gamma t} + X_s \]
\[ X(t) = e^{-\Gamma t} \left( A_+ e^{\Gamma' t} + A_- e^{-\Gamma' t} \right) + X_s \]

where: \( \Gamma = R/2L \) (series) or \( \Gamma = 1/(2RC) \) (parallel);
\[ \omega_o = 1/\sqrt{LC}; \omega' = \sqrt{\omega_o^2 - \Gamma^2}; \text{and} \Gamma' = \sqrt{\Gamma^2 - \omega_o^2} \]
Homework

• HW #33 due today by 4:30 pm in EE 325B
• HW #34 due Fri. – DeCarlo & Lin, Chapter 10:
  – Problem 58
  – Problem 62