ECE 595, Section 10 Numerical Simulations Lecture 15: Beam Propagation Method II

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Outline

- Recap from Monday
- Perfectly Matched Layers
- Finite Elements
- Finite Element BPM
- Reducing FEM Errors

Recap from Monday

- Derivation of Beam Propagation Method
- Nonlinear Schrodinger equation
- Comparison of BPM Strategies
  - FFT
  - Uniform spatial grid
  - Finite element

Recap from Monday

- Beam propagation amounts to solving:

$$\frac{\partial \phi}{\partial z} = (U + W)\phi$$

where:

$$U = \frac{j}{2\beta} \nabla_{\perp}^{2}$$

$$W = \frac{j k_{\perp}^{2}}{2\beta}$$

Perfectly Matched Layers

- In order to prevent lateral reflections (e.g., from PEC boundaries), can introduce perfectly matched layers (PML)
- Several formulations (including split-field and uniaxial), but here we'll follow stretched coordinate PML
- Effected by the transformation:

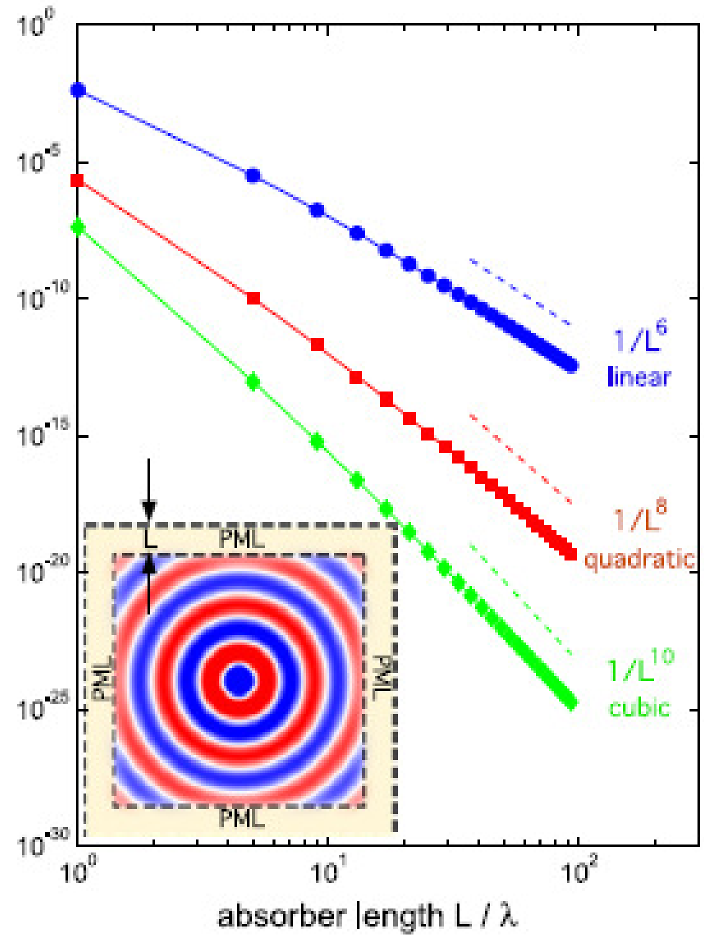
$$\nabla \longrightarrow A \cdot \nabla$$

$$A = \begin{pmatrix} 1 - j\beta & 0 & 0 \\ 0 & 1 - j\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\beta = -\frac{3\lambda\rho^2}{4\pi n d^3} \ln R$$

Perfectly Matched Layers

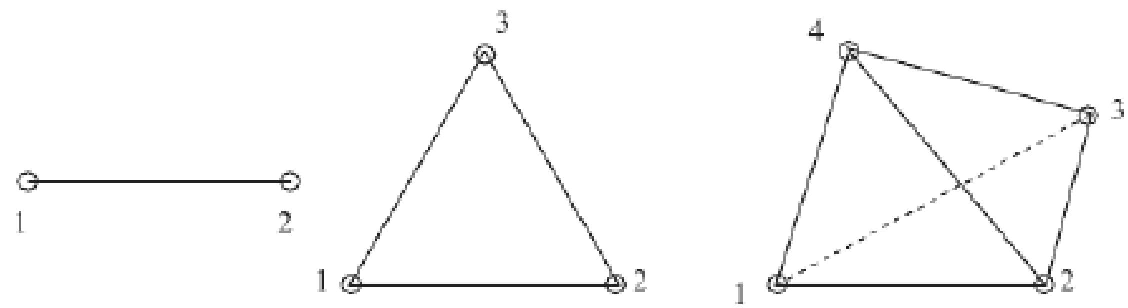
- Residual reflection scales as a power law with PML thickness
- Cubic absorption increase with position offers the best performance



A.F. Oskooi *et al.*, *Comput. Phys. Commun.* (2009)

Finite Elements

- Shapes: 1D, 2D, and 3D



- Shape functions:

$$\text{1D: } u(x) = \alpha + \beta x + \gamma x^2 + \cdots$$

$$\text{2D/3D: } u(x) = \sum_{k=0}^{d} [\alpha_k x^k + \beta_k y^k + \gamma_k z^k]$$

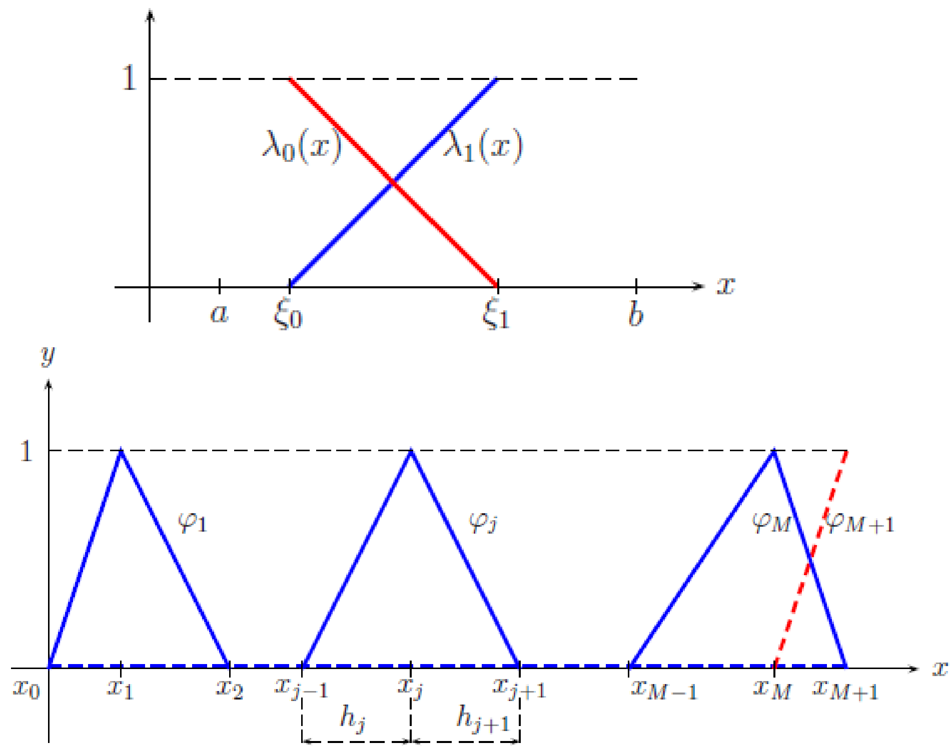
Finite Elements

- Lagrange functions:

$$\lambda_o(x) = \frac{\xi_1 - x}{\xi_1 - \xi_o}$$

$$\lambda_1(x) = \frac{x - \xi_o}{\xi_1 - \xi_o}$$

Basis functions $\varphi_j(x)$ combine the Lagrange functions with compact support



M. Asadzadeh, *Introduction to the Finite Element Method for Differential Equations* (2010)

Finite Element BPM

- In general, can formulate FE problems as:

$$Lu = b$$

  - L is the stiffness matrix, representing overlap between basis functions
  - b is the integral of given PDE with respect to basis
  - u is unknown

Finite Element BPM

- Can define error function as:

$$E = Lu - b$$

- In order to eliminate errors, set weighted residual w_i in test space *v* to zero:

$$\oint_v w_i \,(Lu - b) = 0$$

- Galerkin's method is a specific example of this:

$$\oint_v \psi(Lu - b) = 0$$

where *u*(*x*) are the polynomials we saw earlier

Finite Element BPM

- Can refine accuracy of BPM for wide-angle beam propagation with second derivative in z:

$$\frac{d\zeta}{dz} = -2j\beta\zeta - \nabla_\perp^2 \phi - k_\perp^2 \phi$$

$$\frac{d\phi}{dz} = \zeta$$

- Can then choose a Padé approximant based on initial value of ζ. If ζ(0)=0, then:

$$\zeta = j\beta\left[\sqrt{1 + \frac{\nabla_\perp^2 + k_\perp^2}{\beta^2}} - 1\right]\phi$$

Finite Element BPM

- Applying Galerkin method to second-order BPM equations yields:

$$h_T(x, y, z) = \sum_{j=1}^{N_{px}} h_{xj}(z)\psi_j(x, y)\hat{u}_x + \sum_{j=N_{px}+1}^{N_p} h_{yj}(z)\psi_j(x, y)\hat{u}_y$$

$$[M]\frac{\partial^2\{h_T\}}{\partial z^2} - 2\gamma[M]\frac{\partial\{h_T\}}{\partial z} + ([K] + \gamma^2[M])\{h_T\} = \{0\}$$

$$[M]_{ij} = \int_\Omega \bar{\bar{k}}_a \vec{\psi}_j . \vec{\psi}_i \, d\Omega$$

$$\begin{aligned}[K]_{ij} = &-\int_\Omega (\bar{\bar{k}}_{zz}\nabla_T \times \vec{\psi}_j).(\nabla_T \times \vec{\psi}_i)\, d\Omega + \int_\Omega (\nabla_T \times \vec{\psi}_j)\nabla_T.(\bar{\bar{k}}_b^T \vec{\psi}_i)\, d\Omega \\ &- \oint_{\partial\Omega} (\nabla_T.\vec{\psi}_j)(\bar{\bar{k}}_b^T \vec{\psi}_i).\hat{n}\, d\ell + \int_\Omega \bar{\bar{k}}_c \vec{\psi}_j . \vec{\psi}_i \, d\Omega\end{aligned}$$

Reducing FEM Errors

- Error depends on match between true solution and basis functions
- To reduce error, can try the following:
  - H-adaptivity: decrease the mesh size
  - P-adaptivity: increase the degree of the fitted polynomials
  - HP-adaptivity: combine all of the above

Reducing FEM Errors

- Strategy for reducing errors:
  - Create an initial meshing
  - Compute solution on that meshing
  - Compute the error associated with it
  - If above our tolerance, refine the mesh spacing and start again

Next Class

- Is on Friday, Feb. 15
- Will continue with beam propagation method
- Recommended reading: Obayya, Sections 2.7-2.8