ECE 595, Section 10
Numerical Simulations
Lecture 26: Overview of Transfer Matrix Methods

Prof. Peter Bermel

March 18, 2013

Recap from Fri., Mar. 8

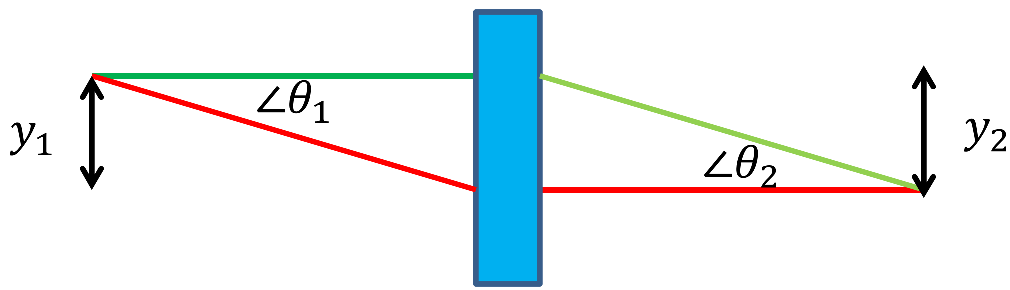
- Periodic Potential Lab
  - Basic principles
  - Input Interface
  - Exemplary Outputs
- CNTbands
  - Basic principles
  - Input Interface
  - Exemplary Outputs

Outline

- Ray-optics transfer matrix
- Wave-optics matrix methods:
  - T-matrix
  - R-matrix
  - S-matrix

Ray Optics Transfer Matrices

- Consider light traveling through an optical element (blue):



- Can capture behavior with 2 rays: ⊥ going in (green); ⊥ going out (red)
- Represent input and output states as ordered pair: (y_k, θ_k)

Ray Optics Transfer Matrices

- Would like to create linear relationship between input and output
- Consider propagation across a distance *d*:

$$y_2 = y_1 + d \tan \theta_1$$

- In paraxial approximation, assume angles are small, such that:

$$\sin \theta \approx \theta$$
$$\tan \theta \approx \theta$$
$$\cos \theta \approx 1 - \theta^2/2$$

Ray Optics Transfer Matrices

- We can now relate input and output states:

$$y_2 = y_1 + d\theta_1$$
$$\theta_1 = \theta_2$$

- Expressed as a matrix:

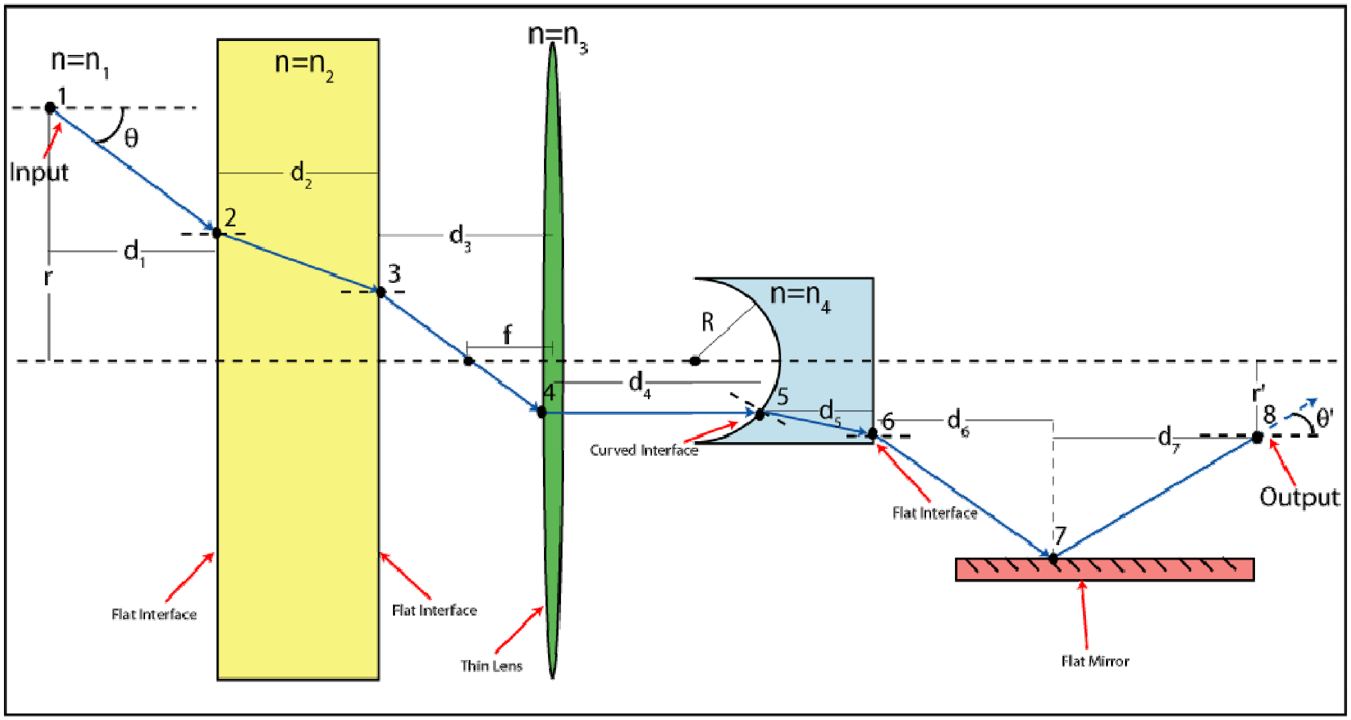
$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

- In general, can write:

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

Ray Optics Transfer Matrices

Example Optical System



- To propagate light through this system:

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & d_7 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_6 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{n_4}{n_1} \end{pmatrix}\begin{pmatrix} 1 & d_5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{n_1 - n_4}{Rn_4} & \dfrac{n_1}{n_4} \end{pmatrix}\begin{pmatrix} 1 & d_4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\cdots\begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

http://www.photonics.byu.edu/ABCD_Matrix_tut.phtml

Wave Optics Transfer Matrices

- If we represent the electric field as two counter-propagating waves:

$$E(x) = E_+ e^{j\beta x} + E_- e^{-j\beta x}$$

- We can use Faraday's law:

$$-\frac{\partial B}{\partial t} = \nabla \times E$$

- To show that:

$$B(x) = \frac{n}{c}\left(E_- e^{-j\beta x} - E_+ e^{j\beta x}\right)$$

Wave Optics Transfer Matrices

- Can calculate each component from total field at *x*=0:

$$E_+ = \frac{1}{2}\left[E(0) - \frac{c}{n}B(0)\right]$$
$$E_- = \frac{1}{2}\left[E(0) + \frac{c}{n}B(0)\right]$$

- Then construct total field at *x*=*L*:

$$E(L) = \frac{1}{2}\left[E(0) - \frac{c}{n}B(0)\right]e^{j\beta L} + \frac{1}{2}\left[E(0) + \frac{c}{n}B(0)\right]e^{-j\beta L}$$

T-Matrices

- Can represent solutions as T-matrices:

$$\begin{bmatrix} E(L) \\ B(L) \end{bmatrix} = \begin{bmatrix} \cos\beta L & -\frac{c}{n}\sin\beta L \\ -\frac{n}{c}\sin\beta L & \cos\beta L \end{bmatrix} \begin{bmatrix} E(0) \\ B(0) \end{bmatrix}$$

- This approach is known as the transfer matrix
- For multiple layers, can take matrix products

Wave Optics Transfer Matrices

- Special case: quarter-wave stack, where $\beta L = \pi/2$:

$$\begin{bmatrix} E(a) \\ B(a) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{c}{n_2} \\ -\frac{n_2}{c} & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{c}{n_1} \\ -\frac{n_1}{c} & 0 \end{bmatrix} \begin{bmatrix} E(0) \\ B(0) \end{bmatrix}$$

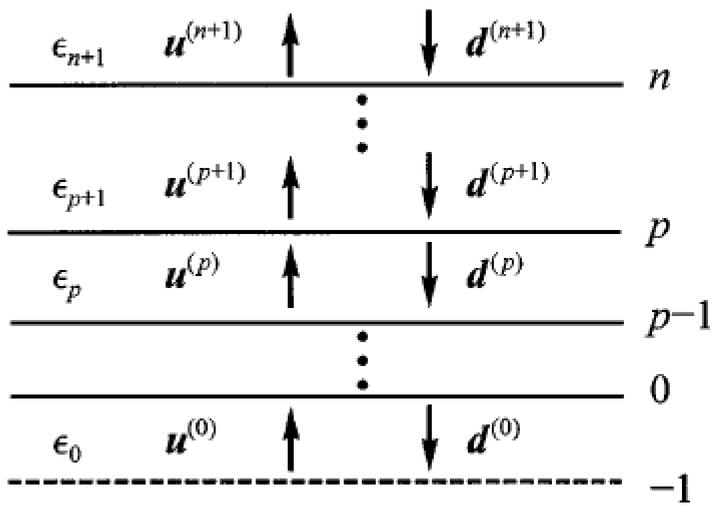
$$\begin{bmatrix} E(a) \\ B(a) \end{bmatrix} = \begin{bmatrix} \frac{n_1}{n_2} & 0 \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} E(0) \\ B(0) \end{bmatrix}$$

- Then transmission for M layers is $T = \left(\frac{n_1}{n_2}\right)^{2M}$

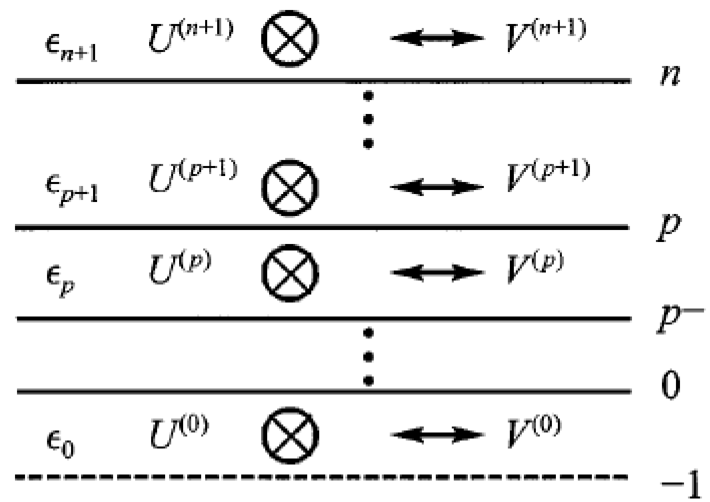
T-Matrices

- Clearly, T-matrices see exponentially growing entries
- A major numerical challenge!
- Can reformulate the problem in a more numerically stable fashion:
  – R-matrix method
  – S-matrix method

S- and R-Matrices



Transfer matrix problem between modes propagating up and down



Transfer matrix problem between two polarizations

Two alternative formulations

S-Matrices

- For S-matrix, connect incoming to outgoing fields from
- Mathematically,

$$\begin{bmatrix} u^{(p+1)} \\ d^{(0)} \end{bmatrix} = \begin{bmatrix} T_{uu}^{(p)} & R_{ud}^{(p)} \\ R_{du}^{(p)} & T_{dd}^{(p)} \end{bmatrix} \begin{bmatrix} u^{(0)} \\ d^{(p+1)} \end{bmatrix}$$

- For input from below:
  – transmission given by $T_{uu}^{(p)}$
  – reflection given by $R_{du}^{(p)}$

R-Matrices

- For R-matrix, connect incoming to outgoing fields from
- Mathematically,

$$\begin{bmatrix} U^{(p+1)} \\ U^{(0)} \end{bmatrix} = \begin{bmatrix} R_{11}^{(p)} & R_{12}^{(p)} \\ R_{21}^{(p)} & R_{22}^{(p)} \end{bmatrix} \begin{bmatrix} V^{(p+1)} \\ V^{(0)} \end{bmatrix}$$

- *U* and *V* can represent *E* and *H* fields; then R-matrix represents field impedance

Next Class

- Is on Wednesday, March 20
- Will continue explaining R- and S-matrices
- Read L. Li, *JOSA A* **13**, 1024-1035