ECE 595, Section 10
Numerical Simulations
Lecture 31: Coupled Mode Theory (CMT)

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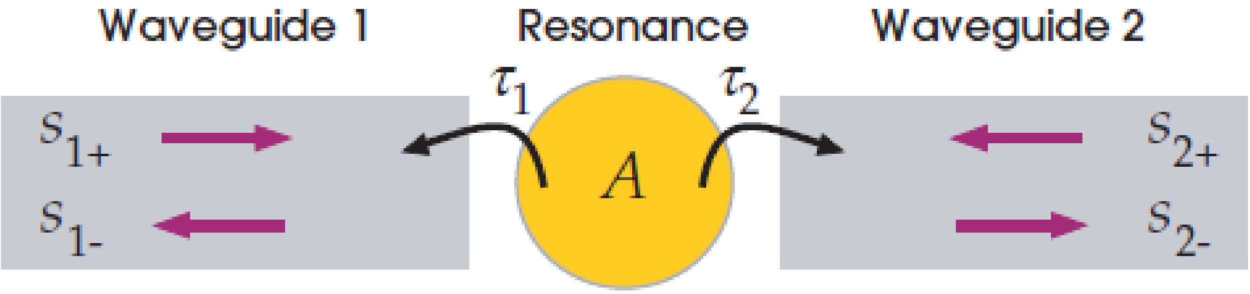
Recap from Wednesday

- Rationale for CAMFR
- Software architecture
- Basic Applications
  - 1D waveguides
  - 2D waveguides
  - 3D cylindrical waveguides
- Advanced Applications
  - Photonic Crystal Splitters
  - VCSELs

Outline

- Recap from Wednesday
- Overview of Coupled Mode Theory
- Derivation of Coupled Mode Equations
- Applications:
  - Single Waveguides
  - Add-Drop filters
  - Waveguide Bends
  - Channel Drop
  - T-Splitters
  - Nonlinear Kerr Waveguides

Coupled-Mode Theory: Basic Concept



H. Haus, *Waves & Fields in Optoelectronics*, Chap. 7 (1984)
W. Suh et al., *IEEE J. Quantum Electron.* **40**, 1511 (2004)
J.D. Joannopoulos et al., *Photonic Crystals*, Chap. 10 (2008).

- Energy exists in 2 forms:
  - Localized resonant modes: $\{A_i\}$
  - Traveling waveguide modes: $\{S_{i+},\ S_{i-}\}$
- Key assumptions:
  - Weak coupling between modes
  - Linearity (i.e., the validity of superposition)
  - Time-reversal symmetry and conservation of energy
  - Time-invariance

Derivation of Coupled Mode Equations

- Assume that:
  - Energy of resonant modes is given by $U_i = |A_i|^2$
  - Incident power of waveguide modes is given by $|S_{i+}|^2$
- Resonator *i* oscillates in phase at frequency ω_i, hence:

$$\frac{dA_i}{dt} = -j\omega_i A_i$$

- Resonator energy decays at rate proportional to energy present:

$$\frac{dU_i}{dt} = -\frac{2U_i}{\tau_i}$$

$$\frac{dA_i}{dt} = -j\omega_i A_i - \frac{A_i}{\tau_i}$$

Derivation of Coupled Mode Equations

- By linearity, coupling of waveguides into modes given by:

$$\frac{dA_i}{dt} = \cdots + \sum_j \alpha_{ij} S_{j+}$$

- For similar reasons, outgoing waveguide modes given by:

$$S_{i-} = \beta_i S_{i+} + \sum_j \gamma_{ij} A_j$$

- By conservation of energy, inputs must be stored or lost:

$$\sum_i \left[|S_{i+}|^2 - |S_{i-}|^2 - \frac{dU_i}{dt} \right] = 0$$

- Special cases can be used to obtain coefficients: $\{\alpha_{ij},\ \beta_i,\ \gamma_{ij}\}$

Derivation of Coupled Mode Equations

- In absence of coupling to resonant modes, conservation of energy requires $|\beta_i| = 1$. Phase depends on convention.
- In absence of input waveguide, must have:

$$0 = |S_{i-}|^2 + \frac{dU_i}{dt}$$

$$0 = |S_{i-}|^2 - \frac{2U_i}{\tau_i}$$

$$0 = |\gamma_i|^2 U_i - \frac{2U_i}{\tau_i}$$

- Thus, $\gamma_i = \sqrt{2/\tau_i}$
- Finally, time reversal implies $\alpha_i = \gamma_i$

Single Waveguide

- For simplest case: 1 waveguide + 1 resonator with 1 input:

$$S_{1-} = S_{1+} - \sqrt{2/\tau_1}A$$

$$\frac{dA}{dt} = -j\omega_o A - \frac{A}{\tau_1} + \sqrt{\frac{2}{\tau_1}}S_{1+}$$

- Transmission can be calculated as quotient:

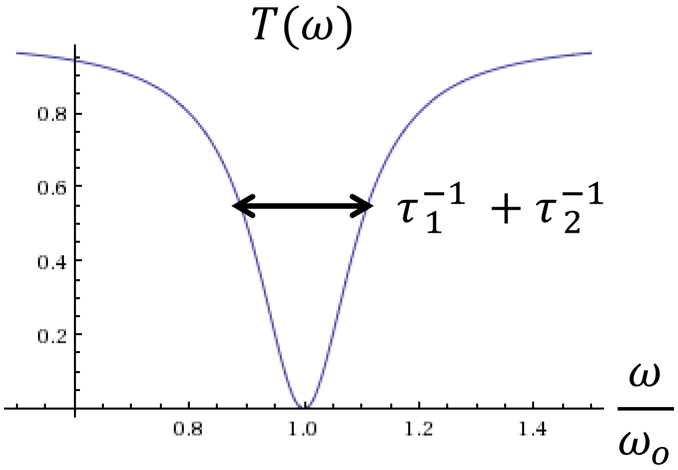
$$R(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \frac{(\omega - \omega_o)^2 + \tau_1^{-2}}{(\omega - \omega_o)^2 + \tau_1^{-2}}$$

- Result: full reflection at all wavelengths, since light has no where else to go!

Application: Add-Drop Filters

- For simple case: 2 waveguides + 1 resonator with 1 input:

$$S_{1-} = S_{1+} - \sqrt{2/\tau_1}A$$
$$S_{2-} = \sqrt{2/\tau_2}A$$
$$\frac{dA}{dt} = -j\omega_o A - \frac{A}{\tau_1} - \frac{A}{\tau_2} + \sqrt{\frac{2}{\tau_1}}S_{1+}$$

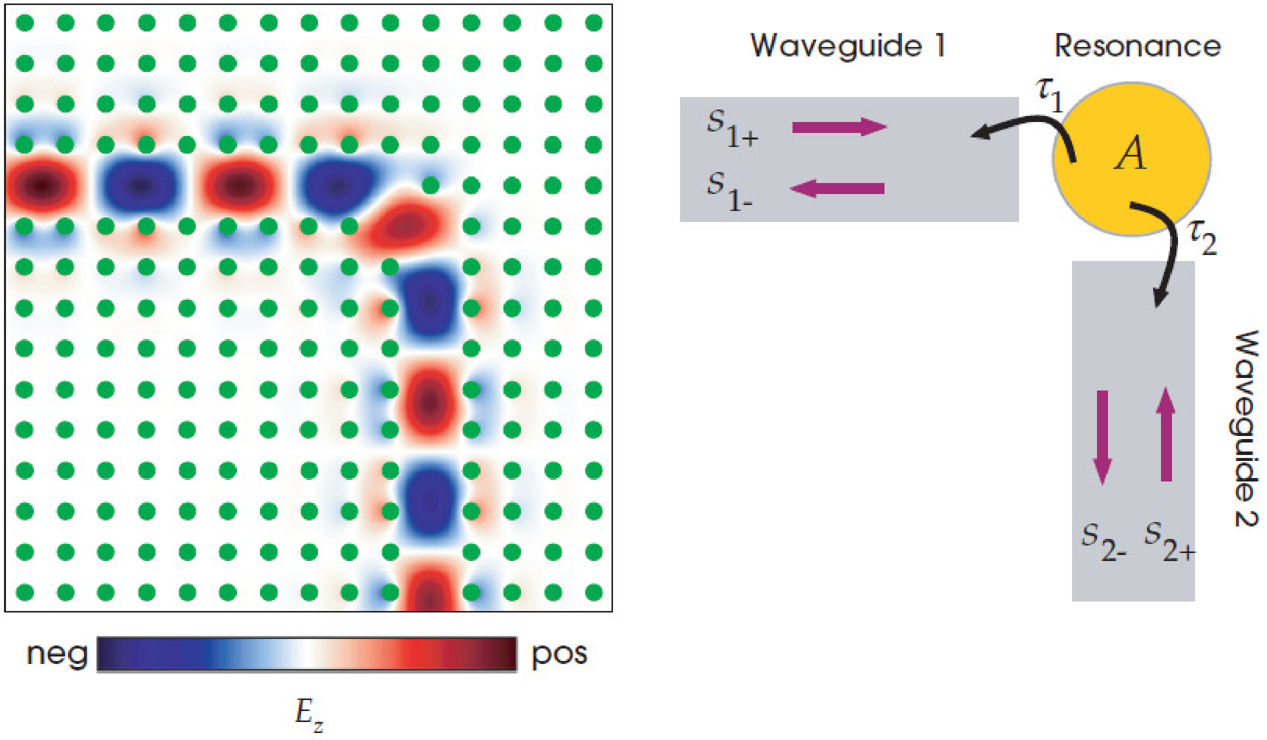


- Transmission can be calculated as quotient:

$$T(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \frac{(\omega - \omega_o)^2 + (\tau_1^{-1} - \tau_2^{-1})^2}{(\omega - \omega_o)^2 + (\tau_1^{-1} + \tau_2^{-1})^2}$$

- Result: a Lorentzian dip in transmission, centered at resonant frequency ω_o

Application: Waveguide Bends



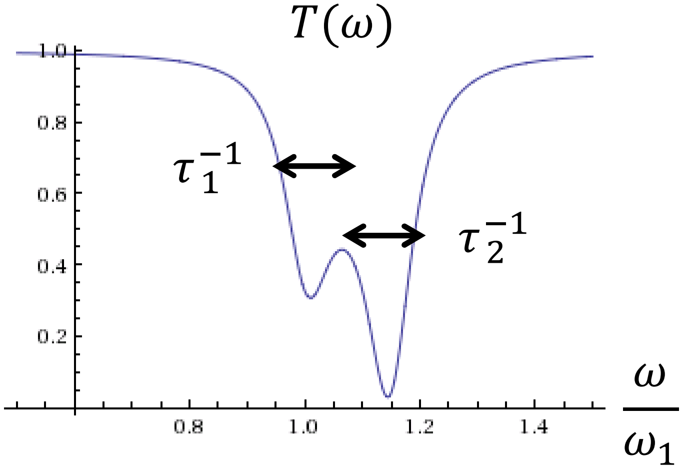
J.D. Joannopoulos *et al*., *Photonic Crystals,* Ch. 10 (Princeton, 2008)

- Can understand a photonic waveguide bend as a special case of the previous problem, with outputs reversed

Application: Channel Drop Filter

- Consider 4 channels + 2 resonators with 1 input:

$$S_{1-} = S_{1+} - \sqrt{2/\tau_1}A_1 - \sqrt{2/\tau_2}A_2$$
$$S_{2-} = \sqrt{2/\tau_1}A_1 + \sqrt{2/\tau_2}A_2$$
$$S_{34-} = \sqrt{2/\tau_3}A_1 + \sqrt{2/\tau_4}A_2$$
$$\frac{dA_1}{dt} = -j\omega_1 A_1 - \sum_i \frac{A_1}{\tau_i} + \sqrt{\frac{2}{\tau_1}}S_{1+}$$

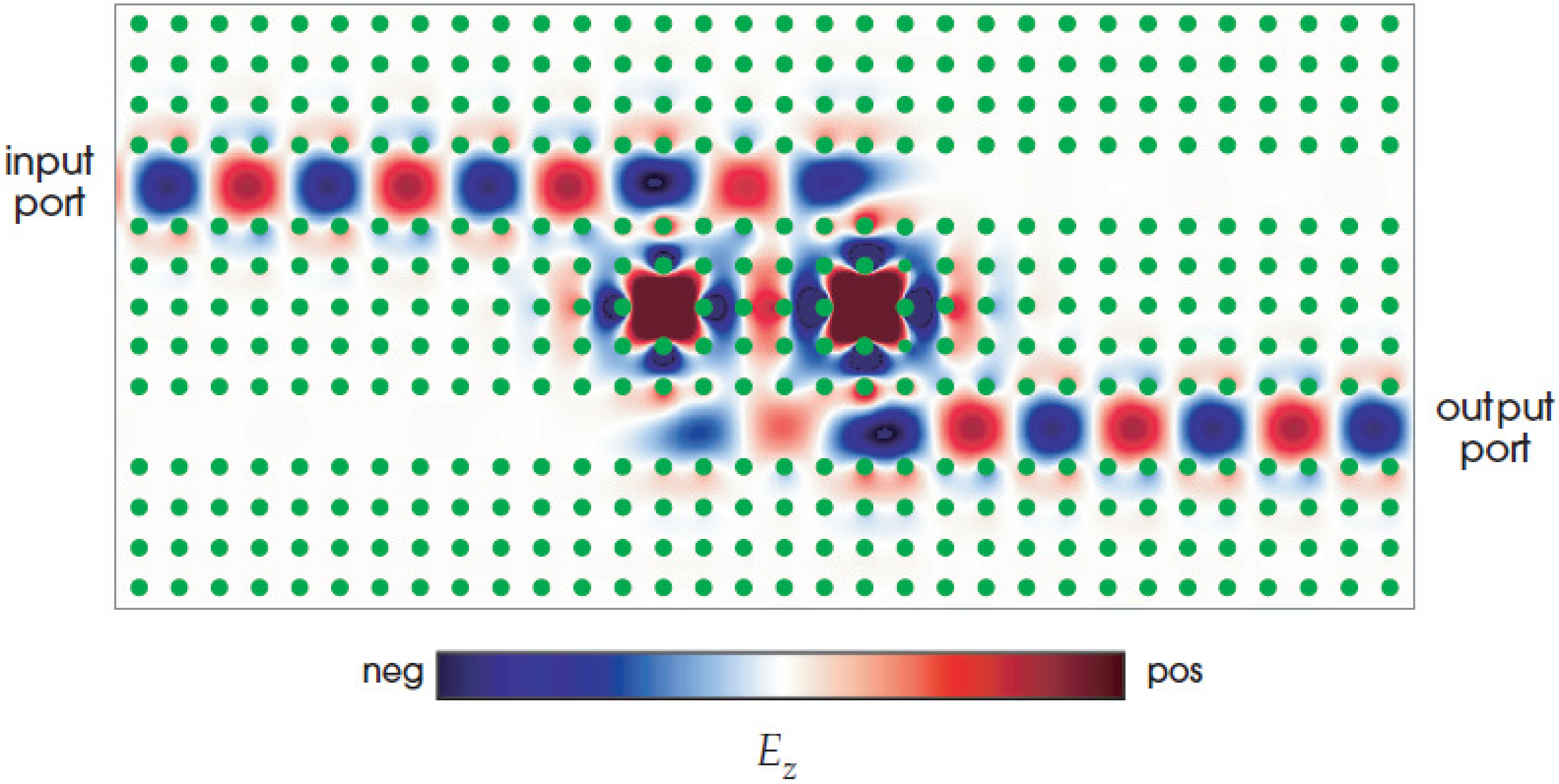


- Transmission can be calculated as quotient:

$$T(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \left|1 - \frac{2/\tau_1}{j(\omega_1 - \omega) + \Gamma} - \frac{2/\tau_2}{j(\omega_2 - \omega) + \Gamma}\right|^2$$

- Result: a Fano lineshape encompassing both resonances ω_1 and ω_2

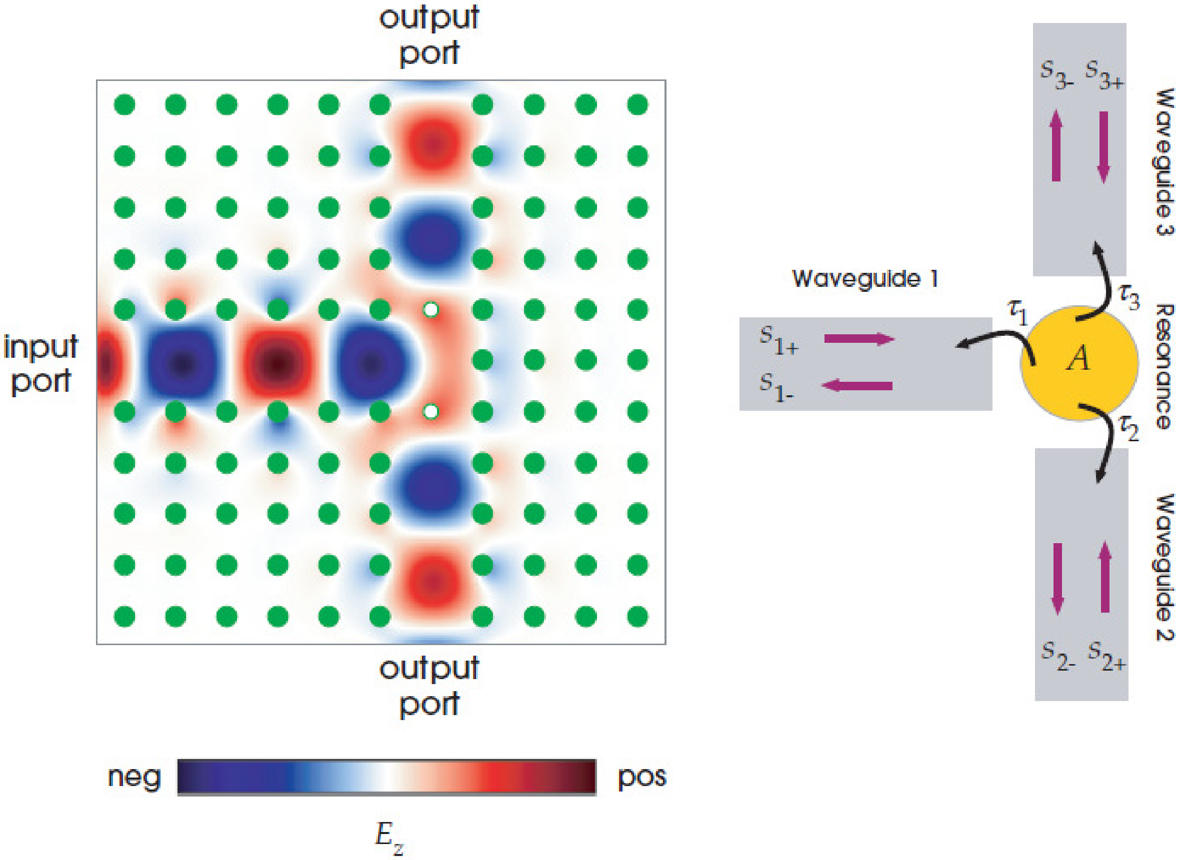
Application: Channel Drop Filter



J.D. Joannopoulos *et al*., *Photonic Crystals,* Ch. 10 (Princeton, 2008)

- Can tune resonator pair to transmit into any desired channel at a target frequency

Application: T-Splitter



J.D. Joannopoulos *et al.*, *Photonic Crystals,* Ch. 10 (Princeton, 2008)

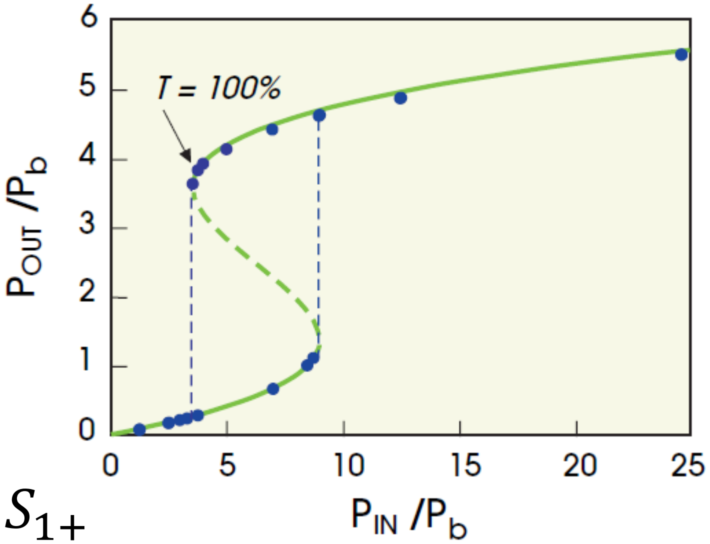
- Can predict coupling strengths needed for perfect forward transmission: $\tau_1^{-1} = \tau_2^{-1} + \tau_3^{-1}$

Application: Kerr Nonlinearities

- Take 2 waveguides + 1 resonator with Kerr nonlinearity and 1 input:

$$S_{1-} = S_{1+} - \sqrt{2/\tau_1}A$$
$$S_{2-} = \sqrt{2/\tau_2}A$$

$$\frac{dA}{dt} = -j(\omega_o + \kappa|A|^2)A - \frac{A}{\tau_1} - \frac{A}{\tau_2} + \sqrt{\frac{2}{\tau_1}}S_{1+}$$



J.D. Joannopoulos *et al.*,
Photonic Crystals, Ch. 10
(Princeton, 2008)

- System becomes *bistable*, with different initial conditions giving rise to different transmission regimes
- Transmission should now be calculated as :

$$T(\omega) = \frac{2}{\tau_2}|A|^2 = \cdots = \frac{1}{1 + (\delta - P_{out}/P_b)^2}$$

Conclusions

- In general, CMT works for a broad range of systems with well-defined and relatively weakly coupled resonances
- Can be readily extended to cases with weak losses, by treating them as additional ‘waveguides’
- Furthermore, in the linear case, most problems can be solved analytically
- Can extend CMT to nonlinear systems (e.g., Kerr media) or time-varying systems, but generally must use ODE solvers to find numerical solutions

Next Class

- Is on Monday, April 1
- Next time: we will discuss coupled mode theory tools: http://nanohub.org/tools/cmtcomb3/