ECE 595, Section 10 Numerical Simulations Lecture 32: Simulations of Coupled Mode Theory (CMT)

Prof. Peter Bermel

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Recap from Friday

- Overview of Coupled Mode Theory
- Derivation of Coupled Mode Equations
- Applications:
  - Single Waveguides
  - Add-Drop filters
  - Waveguide Bends
  - Channel Drop
  - T-Splitters
  - Nonlinear Kerr Waveguides

Outline

- Recap from Friday
- Numerical ODE solvers
  - Initial value problems
  - Boundary value problems
- nanoHUB Tool – CMTcomb3:
  - Rationale
  - Governing ODEs
  - User interface
  - Output analysis

Numerical ODE Solvers

- Objective: to solve ODE (e.g., $\frac{d\boldsymbol{X}}{dt} = \boldsymbol{f}(\boldsymbol{X})$) with greatest accuracy and least computational cost
- Categories:
  - Initial value problems
  - Boundary value problems
- Algorithms:
  - Euler methods
  - Higher-order methods (e.g., Runge-Kutta)
  - Shooting methods
  - Finite element/difference methods

Numerical ODE IVP Solvers

- Euler Method: discretizes original ODE and solves in time steps of Δt:

$$\Delta \boldsymbol{X} = \Delta t \cdot \boldsymbol{f}(\boldsymbol{X})$$

- Advantages: fast, easy to implement
- Disadvantages: inaccurate for many ODEs with modest to large step sizes
- Problem is implicit linear evolution – fails badly for ODEs with significant curvature

Numerical ODE IVP Solvers

- Higher-order methods: incorporate more than just first derivative into solution
- In principle, incorporating nth derivative should reduce errors to $1/\Delta t^{n+1}$
- Example: leapfrog method
  - Let:

$$\frac{d}{dt}\binom{x}{v} = \binom{v}{F(x)}$$

  - Alternate updating x and v:

$$x_{i+1} = x_i + v_i\Delta t + \frac{1}{2}F(x_i)\Delta t^2$$

$$v_{i+1} = v_i + \frac{1}{2}[F(x_i) + F(x_{i+1})]\Delta t$$

Numerical ODE IVP Solvers

- Runge-Kutta methods: incorporate points with spacing less than Δt for each higher accuracy (e.g., ode45 in MATLAB)
- Generally define k_i to incrementally determine slopes within interval:

$$k_i = \Delta t \cdot f(t_n + a_i \Delta t, y_n + \sum_{j=1} b_{ij} k_j)$$

- This gives rise to solutions:

$$y_{n+1} = y_n + \Delta t \sum_{i=1}^{M} c_i k_i$$

- In general, order of solution will be equal to *M*; accuracy will go as $1/\Delta t^{M+1}$

Numerical ODE Solvers

- Stiff solvers: for ODEs with a rapid harmonic oscillation, use backward differentiation formulae:

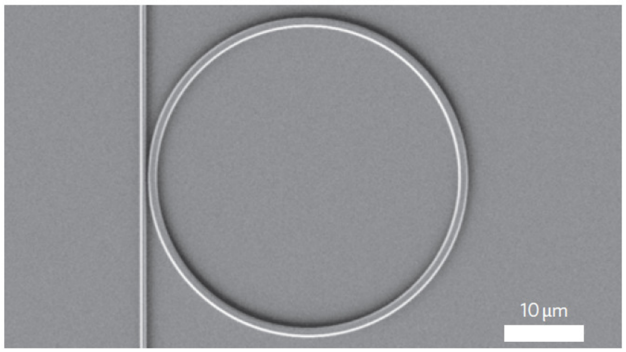
$$\sum_{i=0}^{M} c_i y_{n+i} = \Delta t \cdot f(t_{n+M}, y_{n+M})$$

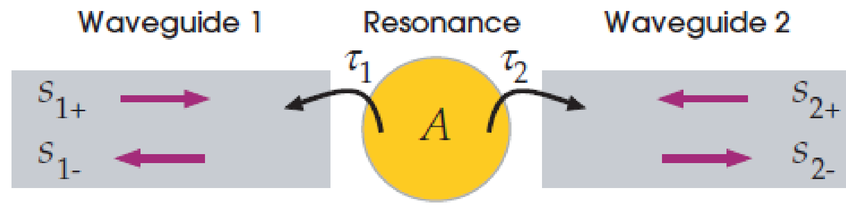
- Convergence can be orders of magnitude better
- Implemented with ode15s in MATLAB

Numerical ODE BVP Solvers

- Shooting method applies Euler method
- However, finite element and finite difference methods are ideal for boundary value problems
- Finite elements: discretize on finite element basis, and solve using Galerkin method
- Finite difference: discretize on grid, and solve using leapfrog method

CMTComb3 – a nanoHUB.org tool





J.D. Joannopoulos et al., *Photonic Crystals*, Chap. 10 (2008).

Goal is to reduce physics of four-wave mixing in a microring resonator to a set of coupled mode equations, then solve them

Recap: Derivation of CMT Equations

- By linearity, coupling of waveguides into modes given by:

$$\frac{dA_i}{dt} = \cdots + \sum_j \alpha_{ij} S_{j+}$$

- For similar reasons, outgoing waveguide modes given by:

$$S_{i-} = \beta_i S_{i+} + \sum_j \gamma_{ij} A_j$$

- By conservation of energy, inputs must be stored or lost:

$$\sum_i \left[|S_{i+}|^2 - |S_{i-}|^2 - \frac{dU_i}{dt} \right] = 0$$

- Special cases can be used to obtain coefficients: $\{\alpha_{ij},\ \beta_i,\ \gamma_{ij}\}$

CMTComb3 – a nanoHUB.org tool

- Basic equations

Loss terms | Nonlinear coupling terms | Pump input

Pump mode:

$$\frac{da_k}{dt} = (i\omega_k - \Gamma_k - \gamma_k)a_k + i\sum_{l,m}^{N} k_{lmk}a_l{}^*a_m a_{k+l-m} + \sqrt{2\Gamma_k}s_+$$

Side modes:

$$\frac{da_k}{dt} = (i\omega_k - \Gamma_k - \gamma_k)a_k + i\sum_{l,m}^{N} k_{lmk}a_l{}^*a_m a_{k+l-m}$$

Output mode:

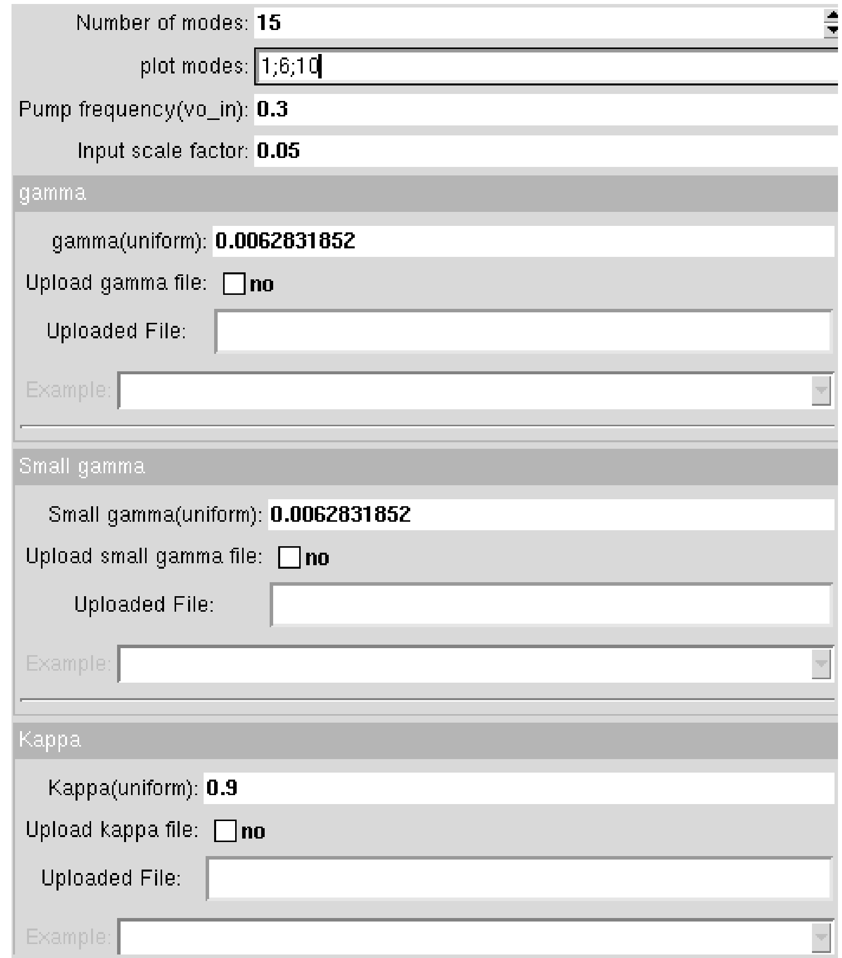
$$\frac{ds_-}{dt} = -\frac{ds_+}{dt} + \sum_{k=1}^{N}\sqrt{2\Gamma_k}\frac{da_k}{dt}$$

R. Shugayev *et al*., "Coupled Mode Theory for Microring Resonator Cavities," presented at *International Workshop for Novel Ideas in Optics* (June 2012, West Lafayette, IN).

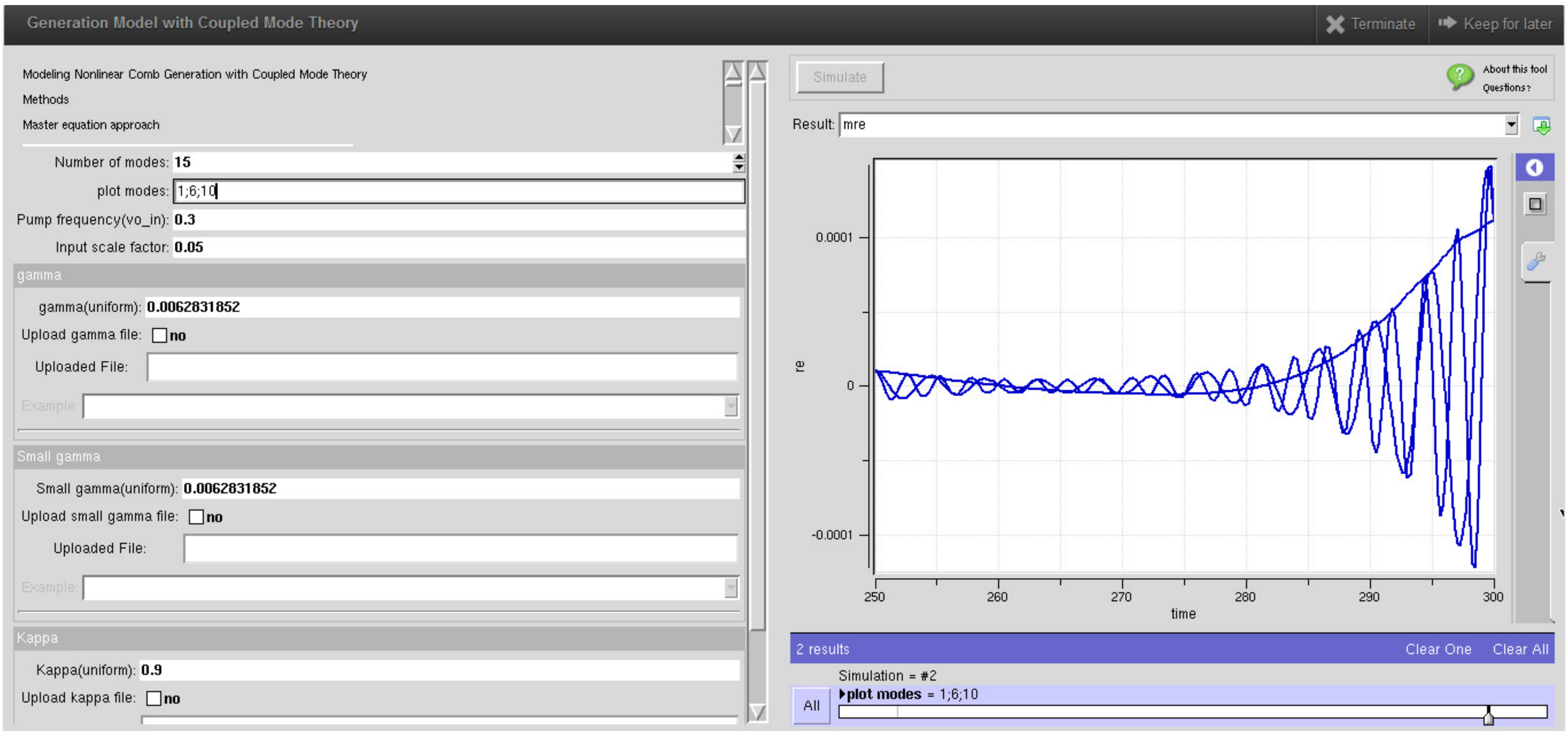
CMTComb3 – a nanoHUB.org tool

Input parameters for user to modify:

- Number of modes
- Plot modes
- Pump frequency
- Input scale factor
- Loss rate Γ
- Nonlinear strength κ



CMTComb3 – a nanoHUB.org tool



CMTComb3 – a nanoHUB.org tool

- Coupled mode theory (CMT) is accurate in the weak-coupling regime, and much faster than full-wave time-domain simulations of comparable systems
- Our CMT comb simulation enables exploration of basic physics and rapid prototyping for specific applications, including metrology and RF signal modulation

Next Class

- Is on Wednesday, April 3
- Next time: we will discuss finite-difference time domain techniques
- Suggested reference: S. Obayya's book, Chapter 5, Sections 4-6