1. **Galerkin’s Method** Consider the following boundary value problem – the Helmholtz equation in 1D, where:

\[
- \frac{\partial^2 E}{\partial x^2} = \left( \frac{n \omega}{c} \right)^2 E. \tag{1}
\]

Let the field vanish at the boundaries, i.e., \( E(0) = E(L) = 0 \), and be unity at the midpoint, i.e., \( E(L/2) = 1 \).

1a. Set the basis function to be quadratic, i.e., \( N_1(x) = x(L - x) \) with a single degree of freedom (i.e., the overall magnitude), and use the basis function \( N_1(x) \) as a testing function \( M_1(x) \) to obtain an approximate solution. The above differential equation can be simplified if we consider integration by parts, or

\[
\int_0^L M_1(x)E''(x)dx = M_1(x)E'(x)|_0^L - \int_0^L M_1'(x)E'(x)dx \tag{2}
\]

1b. How close is this solution to the analytical solution, \( w(x) = \sin(\pi x/L) \)?
2. Basic FEM

2a. Construct the FEM matrix problem associated with the 1D Helmholtz equation presented in the previous problem, using the Galerkin method with the Lagrangian basis functions over 2 elements shown below in Fig. 1.

2b. Solve this matrix problem to find the approximate field values associated with this FEM problem. How close is this to the analytical solution?

\[ N_1(x) = \frac{x-x_2}{x_1-x_2} \]
\[ N_2(x) = \frac{x-x_1}{x_2-x_1} - \frac{x-x_3}{x_2-x_3} \]
\[ N_3(x) = \frac{x-x_2}{x_3-x_2} \]

Figure 1: Linear basis functions used to solve the 1D Helmholtz equation in Problem 2