Outline

• Beam Propagation Method
• Nonlinear Schrodinger Equation
• Fourier space BPM
• Uniform grid BPM with PML
• Finite element formulation
• Vectorial BPM mode solver
Example: Beam Propagation

• Starting from the Helmholtz equation:

\[-\nabla^2 \psi = \left( \frac{n\omega}{c} \right)^2 \psi\]

• One can assume a solution of the form:

\[\psi = \phi e^{-j\beta z}\]

• Where \(\phi\) is slowly varying, which gives rise to:

\[-\nabla^2 \phi + 2j\beta \nabla \phi = k_\perp^2 \psi\]
Beam Propagation

• To simplify problem, drop second derivatives in $z$ – now we can write as:

$$\frac{\partial \phi}{\partial z} = \frac{j}{2\beta} \nabla^2 \phi + \frac{jk^2}{2\beta} \phi$$

• Can simplify by defining two operators:

$$U = \frac{j}{2\beta} \nabla^2 \perp$$

$$W = \frac{jk^2}{2\beta}$$

$$\frac{\partial \phi}{\partial z} = (U + W)\phi$$
Beam Propagation

- For a small $z$-step of size $h$, we can formally write a solution:
  \[ \phi(z + h) = e^{h(U+W)}\phi(z) \]
- If we know that $U$ and $W$ operators commute, we can rewrite as:
  \[ \phi(z + h) = e^{hU}e^{hW}\phi(z) \]
  \[ \phi(z + h) = e^{hU/2}e^{hW}e^{hU/2}\phi(z) \]
Beam Propagation

• Split-step method
  – Propagate half a step with the Laplacian
  – Propagate linear phase shift over the full distance
  – Propagate half a step with the Laplacian
Nonlinear Schrodinger Equation

• Can derive expressions suitable for understanding fibers with dispersion and Kerr nonlinearity:

\[ U = -\frac{j \beta_2}{2} \frac{\partial^2}{\partial t^2} \]

\[ W = -\alpha + \frac{j \kappa}{2} |\phi|^2 \]

\[ \frac{\partial \phi}{\partial z} = (U + W)\phi \]
Nonlinear Schrödinger Equation

• In the presence of nonlinearity, don’t actually know the value of $W(z+h)$

• Can obtain the result iteratively
  – Use $W(z)$ to evaluate $W(z+h)$
  – Work backwards to refine guess for $\phi(z+h)$

• After a few iterations, generally reach a self-consistent solution
BPM Strategies

• Most important decision is handling inhomogeneity well

• Possible strategies:
  – Uniform spatial grid
  – FFT
  – Finite-element method
Uniform Spatial Grid BPM

• Reformulate Laplacian in 2D with:

\[ \nabla^2 \phi \approx \frac{\phi_{i-N} + \phi_{i-1} - 4\phi_i + \phi_{i+1} + \phi_{i+N}}{h^2} \]

• Where \( h \) is the grid spacing
FFT BPM

- Well-suited for diffraction step, where we can rephrase the operator as:

\[ U = -\frac{j}{2\beta} (k + G) \frac{2}{\perp} \]

- Can transform before and then back afterwards, via FFT, as in MPB solver
Example: Beam Propagation

\[
[xx, yy] = \text{meshgrid}([xa:del:xb-del], [1:1:zmax]);
\]

\[
\text{mode} = A*\exp(-((x+x0)/WO).^2); \quad \% \text{Gaussian pulse}
\]

\[
\text{dftmode} = \text{fix}(\text{fft(mode)}); \quad \% \text{DFT of Gaussian pulse}
\]

\[
\text{zz} = \text{imread('ybranch.bmp','BMP');} \quad \% \text{Upload image with the profile}
\]

\[
\text{phase1} = \exp((i*deltaz*kx.^2)./(nbar*k0 + \sqrt{\text{max}(0,nbar^2*k0*2 - kx.^2)}));
\]

\[
\text{for } k = 1:zmax,
\]

\[
\quad \text{phase2} = \exp(-(od + i*(n(k,:) - nbar)*k0)*deltaz);
\]

\[
\quad \text{mode} = \text{ifft}((\text{fft(mode)}.*\text{phase1})).*\text{phase2};
\]

\[
\quad \text{zz}(k,:) = \text{abs(mode)};
\]

\text{end}
Example: Beam Propagation
Perfectly Matched Layers

• In order to prevent lateral reflections (e.g., from PEC boundaries), can introduce perfectly matched layers (PML)

• Several formulations (including split-field and uniaxial), but here we’ll follow stretched coordinate PML

• Effected by the transformation:

\[
\nabla \rightarrow A \cdot \nabla
\]

\[
A = \begin{pmatrix}
1 - j\beta & 0 & 0 \\
0 & 1 - j\beta & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\beta = -\frac{3\lambda \rho^2}{4\pi n d^3} \ln R
\]
Perfectly Matched Layers

- Residual reflection scales as a power law with PML thickness
- Cubic absorption increase with position offers the best performance

Finite Element BPM

- Finite element method consists of dividing a spatial domain in 1D, 2D or 3D into a mesh
- Mesh generally has D+1 vertices
- Solution can take various forms, but usually a tent function within each D+1-gon
Finite Elements

• Shapes: 1D, 2D, and 3D

• Shape functions:

\[
1D: u(x) = \alpha + \beta x + \gamma x^2 + \cdots
\]

\[
2D/3D: u(x) = \sum_{k=0}^{d} [\alpha_k x^k + \beta_k y^k + \gamma_k z^k]
\]
Finite Elements

- Lagrange functions:
  \[ \lambda_0(x) = \frac{\xi_1 - x}{\xi_1 - \xi_0} \]
  \[ \lambda_1(x) = \frac{x - \xi_0}{\xi_1 - \xi_0} \]

Basis functions \( \varphi_j(x) \) combine the Lagrange functions with compact support.

Finite Element BPM

• In general, can formulate FE problems as:
  \[ Lu = b \]
  – \( L \) is the stiffness matrix, representing overlap between basis functions
  – \( b \) is the integral of given PDE with respect to basis
  – \( u \) is unknown

• Value of FEA comes from:
  – Spatial flexibility: can define each element to vary in size quite substantially
  – Speed: properly chosen basis functions have compact support, leading to a sparse matrix
Finite Element BPM

• Can define error function as:

\[ E = Lu - b \]

• In order to eliminate errors, set weighted residual \( w_i \) in test space \( v \) to zero:

\[ \int_v w_i (Lu - b) = 0 \]

• Galerkin’s method is a specific example of this:

\[ \int_v \psi (Lu - b) = 0 \]

where \( u(x) \) are the polynomials we saw earlier
Finite Element BPM

- Can refine accuracy of BPM for wide-angle beam propagation with second derivative in z:

\[
\frac{d\zeta}{dz} = -2j\beta\zeta - \nabla_{\perp}^2 \phi - k_{\perp}^2 \phi
\]

\[
\frac{d\phi}{dz} = \zeta
\]

- Can then choose a Padé approximant based on initial value of \( \zeta \). If \( \zeta(0)=0 \), then:

\[
\zeta = j\beta \left[ \sqrt{1 + \frac{\nabla_{\perp}^2 + k_{\perp}^2}{\beta^2}} - 1 \right] \phi
\]
Finite Element BPM

- Applying Galerkin method to second-order BPM equations yields:

\[
\begin{align*}
h_T(x, y, z) &= \sum_{j=1}^{N_{px}} h_{xj}(z)\psi_j(x, y)\hat{u}_x + \sum_{j=N_{px}+1}^{N_p} h_{yj}(z)\psi_j(x, y)\hat{u}_y \\
\frac{\partial^2}{\partial z^2} [M] \{h_T\} - 2\gamma [M] \frac{\partial}{\partial z} \{h_T\} + ([K] + \gamma^2 [M]) \{h_T\} &= \{0\} \\
[M]_{ij} &= \int_{\Omega} \tilde{k}_a \tilde{\psi}_j \cdot \tilde{\psi}_i d\Omega \\
[K]_{ij} &= -\int_{\Omega} (\tilde{k}_{zz} \nabla_T \times \tilde{\psi}_j) \cdot (\nabla_T \times \tilde{\psi}_i) d\Omega + \int_{\Omega} (\nabla_T \times \tilde{\psi}_j) \nabla_T \cdot (\tilde{k}_{b} \tilde{\psi}_i) d\Omega \\
&\quad - \int_{\partial \Omega} (\nabla_T \cdot \tilde{\psi}_j) (\tilde{k}_{b} \tilde{\psi}_i) \cdot \hat{n} d\ell + \int_{\Omega} \tilde{k}_c \tilde{\psi}_j \cdot \tilde{\psi}_i d\Omega
\end{align*}
\]
Reducing FEM Errors

• Error depends on match between true solution and basis functions

• To reduce error, can try the following:
  – H-adaptivity: decrease the mesh size
  – P-adaptivity: increase the degree of the fitted polynomials
  – HP-adaptivity: combine all of the above
Reducing FEM Errors

• Strategy for reducing errors:
  – Create an initial meshing
  – Compute solution on that meshing
  – Compute the error associated with it
  – If above our tolerance, refine the mesh spacing and start again
BPM Mode Solver

• Can extend BPM method to solve for modes, by propagating in the imaginary direction

• First, drop all derivatives in BPM equation:

\[ [K]\{h_{t,l}\} = -\gamma^2[M]\{h_{t,l}\} \]

• Second, write down next step in z:

\[ \{h_{t,l}\}_{k+1} = \frac{-2\gamma - 0.5\Delta z k_o^2 (n_{eff,\ell}^2 - n_o^2)}{-2\gamma + 0.5\Delta z k_o^2 (n_{eff,\ell}^2 - n_o^2)} \{h_{t,l}\}_k \]

• Third, substitute special value of \( \Delta z \):

\[ \Delta z \approx j \frac{4n_o}{(n_{eff,\ell}^2 - n_o^2)k_o} \]
BPM Mode Solver

• Since $\Delta z$ initially unknown, assume largest index possible, and decrease it as needed

• Will eventually converge to correct answer and effective refractive index:

$$ n_{\text{eff}, \ell,k}^2 = \frac{\{h_1\}^* \{K\}_k \{h_1\}_k}{k_o^2 \{h_1\}^*_k \{M\}_k \{h_1\}_k} $$

• Can use Gram-Schmidt normalization procedure to find higher-order modes:

$$ \{h_{t,1}\}_{\text{new}} = \{h_{t,1}\} - \sum_{\ell=1}^{i-1} \frac{\{h_{t,\ell}\}^* \{M\} \{h_{t,1}\}_1 \{h_{t,\ell}\}}{\{h_{t,\ell}\}_1^* \{M\} \{h_{t,\ell}\}} \{h_{t,\ell}\} $$
VBPM on a Waveguide: Problem Description

- Cross section defined above; \( \lambda = 1.3 \ \mu m \)
- Propagation along \( z \) is semi-infinite
- Must grid space with first-order triangular elements in cross-sectional plane; choose PML to reduce reflections to \( 10^{-100} \)
- Will vary \( \Delta z \) for maximum effectiveness
VBPM on a Waveguide

- Fundamental mode is calculated accurately with 12,800 first-order triangular elements
VBPM on a Waveguide

- Propagation step size in $Z$, known as $\Delta Z$, should equal transverse dimensions for best accuracy
VBPM on a Waveguide: Longitudinal Imaginary Propagation

- With optimal step size, can solve the fundamental mode of both polarizations in a pretty modest number of steps!
VBPM on a Waveguide: Accuracy

- Accuracy of calculation of waveguide coupling length as a function of mesh divisions $N$
VBPM on a Waveguide

- Accuracy of coupling length as a function of $\Delta Z$ saturates below one wavelength.
VBPM on a Photonic Crystal Fiber

- Originally conceived of by P.J. Russell
- Confines light to core without total internal reflection!
VBPM on a PhC Fiber

• Effective index vs. PhC period
VBPM on a PhC Fiber

- $H_y$ field distributions for the fundamental TE modes
VBPM on a PhC Fiber

• Confinement loss decreases sharply as period $\Lambda$ increases
VBPM on a PhC Fiber

- Variation of the effective mode area with PhC period $\Lambda$

![Graph showing variation of effective mode area with period $\Lambda$](image)
VBPM on a PhC Fiber

- Effective index increases modestly with increasing period $\Lambda$, indicating increased mode confinement
VBPM on a PhC Fiber

- Calculated dispersion relation (effective index versus wavelength) for a PhC Fiber
• Obtained dispersion $D = d^2 k / d\omega^2$ from earlier data
• Note modest changes in parameters flip sign of $D$
Next Class

• Will discuss beam propagation method
• Recommended reading: Obayya, Chapter 2