# ECE 695
# Numerical Simulations
# Lecture 19: Transfer Matrices

Prof. Peter Bermel

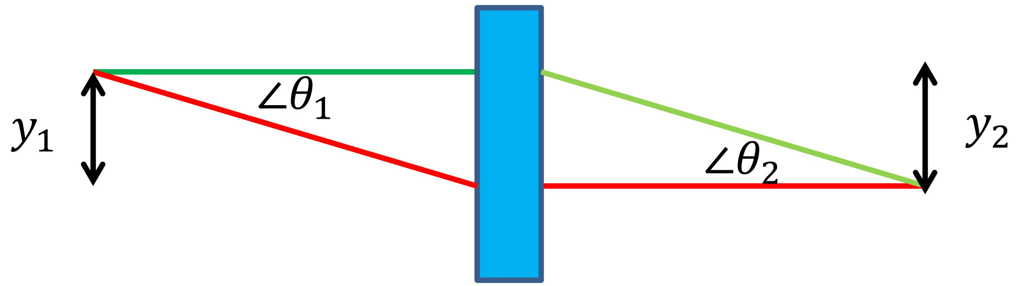
February 27, 2017

# Outline

- Ray-optics transfer matrix
- Wave-optics matrix methods:
  - T-matrix
  - R-matrix
  - S-matrix
- Periodic solution strategy
  - Stacked periodicity
  - Transverse periodicity

# Ray Optics Transfer Matrices

- Consider light traveling through an optical element (blue):




- Can capture behavior with 2 rays: ⊥ going in (green); ⊥ going out (red)
- Represent input and output states as ordered pair: $(y_k, \theta_k)$

# Ray Optics Transfer Matrices

- Would like to create linear relationship between input and output
- Consider propagation across a distance *d*:

$$y_2 = y_1 + d\tan\theta_1$$

- In paraxial approximation, assume angles are small, such that:

$$\sin\theta \approx \theta$$

$$\tan\theta \approx \theta$$

$$\cos\theta \approx 1 - \theta^2/2$$

# Ray Optics Transfer Matrices

- We can now relate input and output states:

$$y_2 = y_1 + d\theta_1$$
$$\theta_1 = \theta_2$$

- Expressed as a matrix:

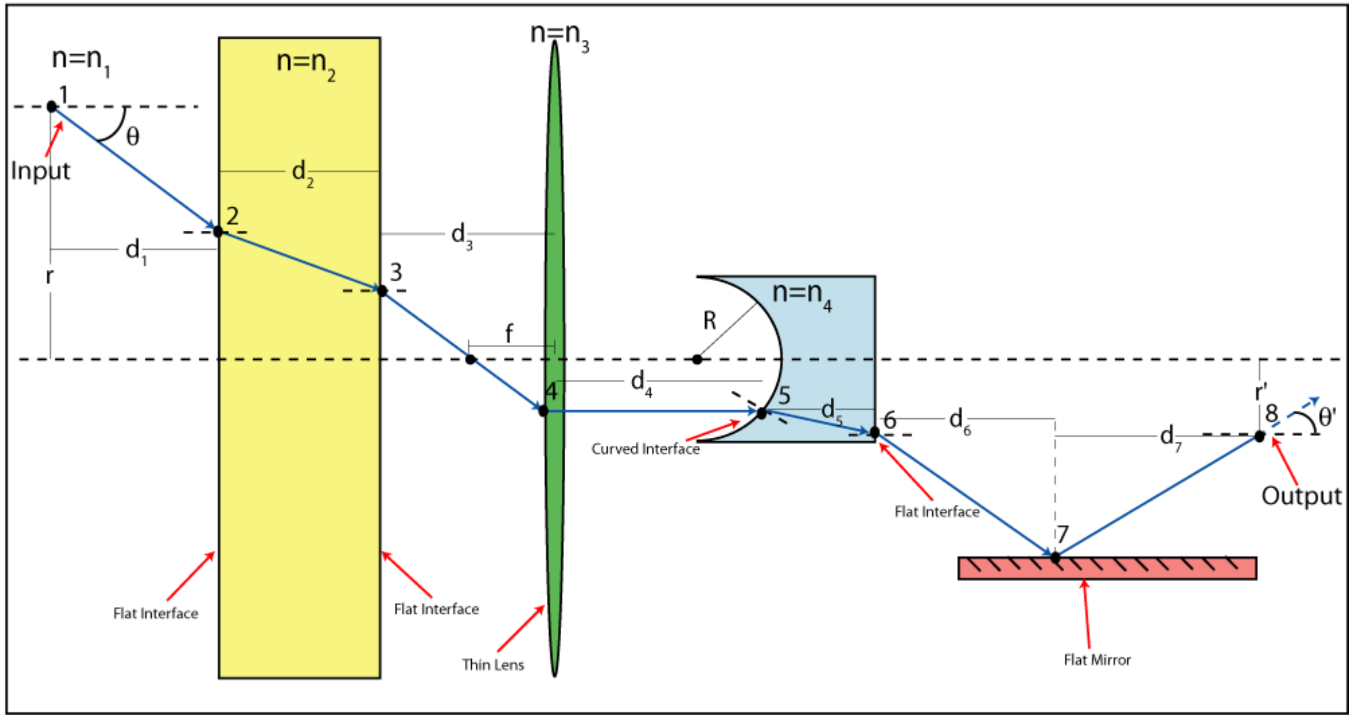
$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

- In general, can write:

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

# Ray Optics Transfer Matrices

Example Optical System




- To propagate light through this system:

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & d_7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d_6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_4}{n_1} \end{pmatrix} \begin{pmatrix} 1 & d_5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_4}{Rn_4} & \frac{n_1}{n_4} \end{pmatrix} \begin{pmatrix} 1 & d_4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \cdots \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

http://www.photonics.byu.edu/ABCD_Matrix_tut.phtml

# Wave Optics Transfer Matrices

- If we represent the electric field as two counter-propagating waves:

$$E(x) = E_+ e^{j\beta x} + E_- e^{-j\beta x}$$

- We can use Faraday's law:

$$-\frac{\partial B}{\partial t} = \nabla \times E$$

- To show that:

$$B(x) = \frac{n}{c}\left(E_- e^{-j\beta x} - E_+ e^{j\beta x}\right)$$

# Wave Optics Transfer Matrices

- Can calculate each component from total field at *x*=0:

$$E_+ = \frac{1}{2}\left[E(0) - \frac{c}{n}B(0)\right]$$
$$E_- = \frac{1}{2}\left[E(0) + \frac{c}{n}B(0)\right]$$

- Then construct total field at *x*=*L*:

$$E(L) = \frac{1}{2}\left[E(0) - \frac{c}{n}B(0)\right]e^{j\beta L} + \frac{1}{2}\left[E(0) + \frac{c}{n}B(0)\right]e^{-j\beta L}$$

# T-Matrices

- Can represent solutions as T-matrices:

$$\begin{bmatrix} E(L) \\ B(L) \end{bmatrix} = \begin{bmatrix} \cos\beta L & -j\dfrac{c}{n}\sin\beta L \\ -j\dfrac{n}{c}\sin\beta L & \cos\beta L \end{bmatrix} \begin{bmatrix} E(0) \\ B(0) \end{bmatrix}$$

- This approach is known as the transfer matrix
- For multiple layers, can take matrix products

# Wave Optics Transfer Matrices

- Special case: quarter-wave stack, where $\beta L = \pi/2$:

$$\begin{bmatrix} E(a) \\ B(a) \end{bmatrix} = \begin{bmatrix} 0 & -j\dfrac{c}{n_2} \\ -j\dfrac{n_2}{c} & 0 \end{bmatrix} \begin{bmatrix} 0 & -j\dfrac{c}{n_1} \\ -j\dfrac{n_1}{c} & 0 \end{bmatrix} \begin{bmatrix} E(0) \\ B(0) \end{bmatrix}$$

$$\begin{bmatrix} E(a) \\ B(a) \end{bmatrix} = -\begin{bmatrix} \dfrac{n_1}{n_2} & 0 \\ 0 & \dfrac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} E(0) \\ B(0) \end{bmatrix}$$

- Then power transmission for *M* periods is $T = \left(\frac{n_1}{n_2}\right)^{2M}$

# T-Matrix Construction

- Layer $t$-matrix consists of product of two terms:
$$\tilde{t}^{(p)} = t^{(p)}\phi^{(p)}$$
- Interface $t$-matrix:
$$t^{(p)} = \frac{W^{(p)}}{W^{(p+1)}}$$
- Propagation matrix:
$$\phi^{(p)} = \begin{bmatrix} \exp\left(j\beta_{m,+}^{(p)}\Delta y_p\right) & 0 \\ 0 & \exp\left(j\beta_{m,-}^{(p)}\Delta y_p\right) \end{bmatrix}$$

# T-Matrix Construction

- In the special case of a quarter wave stack, you obtain:

$$\tilde{t}^{(p)} = \frac{1}{2n_2}\begin{bmatrix} n_2 + n_1 & n_2 - n_1 \\ n_2 - n_1 & n_2 + n_1 \end{bmatrix}\begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}$$

- For one period, this becomes

$$\tilde{t}^{(p)} = -\frac{1}{4n_1 n_2}\begin{bmatrix} n_2 + n_1 & n_1 - n_2 \\ n_2 - n_1 & -n_2 - n_1 \end{bmatrix}\begin{bmatrix} n_2 + n_1 & n_2 - n_1 \\ n_1 - n_2 & -n_2 - n_1 \end{bmatrix}$$

$$= -\frac{1}{2n_1 n_2}\begin{bmatrix} n_1^2 + n_2^2 & n_2^2 - n_1^2 \\ n_2^2 - n_1^2 & n_1^2 + n_2^2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

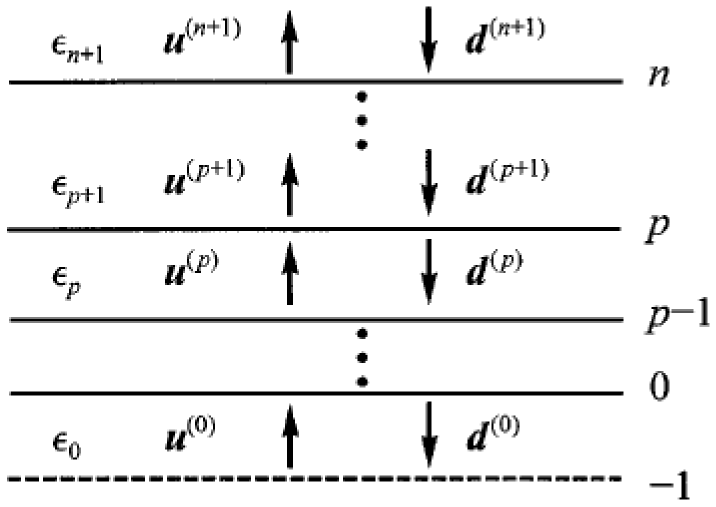
- For *M* periods, just exponentiate to obtain:

$$\tilde{t}^{(p)} = a^M + Ma^{M-1}b$$

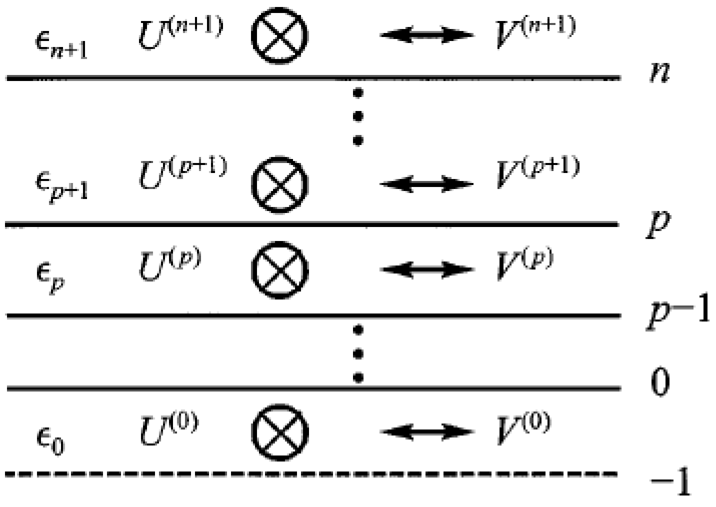
# T-Matrices

- Clearly, T-matrices see exponentially growing entries
- A major numerical challenge!
- Can reformulate the problem in a more numerically stable fashion:
  - R-matrix method
  - S-matrix method

# S- and R-Matrices




Transfer matrix problem between modes propagating up and down




Transfer matrix problem between two polarizations

## Two alternative formulations

# R-Matrices

- For R-matrix, connect polarizations U on both sides to polarizations V on both sides
- Mathematically,

$$\begin{bmatrix} U^{(p+1)} \\ U^{(0)} \end{bmatrix} = \begin{bmatrix} R^{(p)}_{11} & R^{(p)}_{12} \\ R^{(p)}_{21} & R^{(p)}_{22} \end{bmatrix} \begin{bmatrix} V^{(p+1)} \\ V^{(0)} \end{bmatrix}$$

- *U* and *V* can represent *E* and *H* fields; then R-matrix represents field impedance

# S-Matrices

- For S-matrix, connect incoming to outgoing fields from boundaries of region
- Mathematically,

$$\begin{bmatrix} u^{(p+1)} \\ d^{(0)} \end{bmatrix} = \begin{bmatrix} T_{uu}^{(p)} & R_{ud}^{(p)} \\ R_{du}^{(p)} & T_{dd}^{(p)} \end{bmatrix} \begin{bmatrix} u^{(0)} \\ d^{(p+1)} \end{bmatrix}$$

- For input from below:
  - transmission from bottom to top given by $T_{uu}^{(p)}$
  - reflection at bottom given by $R_{du}^{(p)}$

# S-Matrices: Key Properties

- In absence of loss, S matrices are unitary, i.e.: $S^{\dagger}S = 1$
- Result is no exponentially growing terms – mathematically, all exponentials in denominators
- Greater numerical stability vis-a-vis *T*-matrices

# S-Matrix Construction

- If we already know answer from T-matrix:

$$\begin{bmatrix} u^{(p+1)} \\ d^{(p+1)} \end{bmatrix} = \begin{bmatrix} A^{(p)} & B^{(p)} \\ C^{(p)} & D^{(p)} \end{bmatrix} \begin{bmatrix} u^{(0)} \\ d^{(0)} \end{bmatrix}$$

- Then we can rearrange equations:

$$D^{(p)}\, d^{(0)} = d^{(p+1)} - C^{(p)} u^{(0)}$$

$$u^{(p+1)} = \left[A^{(p)} - B^{(p)} C^{(p)} / D^{(p)}\right] u^{(0)} + \left[B^{(p)} / D^{(p)}\right] d^{(p+1)}$$

- Thus, we have:

$$\begin{bmatrix} T^{(p)}_{uu} & R^{(p)}_{ud} \\ R^{(p)}_{du} & T^{(p)}_{dd} \end{bmatrix} = \begin{bmatrix} A^{(p)} - B^{(p)} C^{(p)} / D^{(p)} & B^{(p)} / D^{(p)} \\ -C^{(p)} / D^{(p)} & 1 / D^{(p)} \end{bmatrix}$$

# S-Matrix Construction

- For p layers, can construct p+1th layer using the following relations:
- First, match fields with boundary conditions:

$$W^{(p+1)} \begin{bmatrix} u^{(p+1)}(y_p) \\ d^{(p+1)}(y_p) \end{bmatrix} = W^{(p)} \begin{bmatrix} u^{(p)}(y_p) \\ d^{(p)}(y_p) \end{bmatrix}$$

- Second, construct layer *t*-matrix:

$$\begin{bmatrix} u^{(p+1)}(y_p) \\ d^{(p+1)}(y_p) \end{bmatrix} = \tilde{t}^{(p)} \begin{bmatrix} u^{(p)}(y_{p-1}) \\ d^{(p)}(y_{p-1}) \end{bmatrix}$$

# S-Matrix Construction

- By analogy, can define s-layer matrix such that:

$$\begin{bmatrix} u^{(p+1)}(y_p) \\ d^{(p)}(y_{p-1}) \end{bmatrix} = \tilde{s}^{(p)} \begin{bmatrix} u^{(p)}(y_{p-1}) \\ d^{(p+1)}(y_p) \end{bmatrix}$$

- Can then express in terms of s-interface matrix:

$$\tilde{s}^{(p)} = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(-j\beta_{m,-}^{(p)}\,\Delta y_p\right) \end{bmatrix} s^{(p)} \begin{bmatrix} \exp\left(j\beta_{m,+}^{(p)}\,\Delta y_p\right) & 0 \\ 0 & 1 \end{bmatrix}$$

# S-Matrix Construction

- We can relate the *s*-interface matrix to the *t*-interface matrix from before:

$$s^{(p)} = \begin{bmatrix} t_{11}^{(p)} - t_{12}^{(p)} t_{22}^{(p)\,-1} t_{21}^{(p)} & t_{12}^{(p)} t_{22}^{(p)\,-1} \\ -t_{22}^{(p)\,-1} t_{21}^{(p)} & t_{22}^{(p)\,-1} \end{bmatrix}$$

- Then iteratively construct next S-matrix via:

$$\begin{bmatrix} T_{uu}^{(p)} & R_{ud}^{(p)} \\ R_{du}^{(p)} & T_{dd}^{(p)} \end{bmatrix} = \begin{bmatrix} \tilde{t}_{uu}^{(p)} \left[1 - R_{ud}^{(p-1)} \tilde{r}_{du}^{(p)}\right]^{-1} T_{uu}^{(p-1)} & \tilde{r}_{ud}^{(p)} + \tilde{t}_{uu}^{(p)} R_{ud}^{(p-1)} \left[1 - \tilde{r}_{du}^{(p)} R_{ud}^{(p-1)}\right]^{-1} \tilde{t}_{dd}^{(p)} \\ R_{ud}^{(p-1)} + T_{dd}^{(p-1)} \tilde{r}_{du}^{(p)} \left[1 - R_{ud}^{(p-1)} \tilde{r}_{du}^{(p)}\right]^{-1} T_{uu}^{(p-1)} & T_{dd}^{(p-1)} \left[1 - \tilde{r}_{du}^{(p)} R_{ud}^{(p-1)}\right]^{-1} \tilde{t}_{dd}^{(p)} \end{bmatrix}$$

# S-Matrices: Periodicity

- What happens if one or more layers are periodic?
- Then there are two types of coupling:
  - Layer-to-layer (refractive)
  - Mode-to-mode (diffractive)
- Can both be treated in a single framework?

# S-Matrices: Stacked Periodicity

- Recall that:

$$s^{(p)} = \begin{bmatrix} t_{11}^{(p)} - t_{12}^{(p)} t_{22}^{(p)\,-1} t_{21}^{(p)} & t_{12}^{(p)} t_{22}^{(p)\,-1} \\ -t_{22}^{(p)\,-1} t_{21}^{(p)} & t_{22}^{(p)\,-1} \end{bmatrix}$$

- For quarter wave stack, express in terms of t-matrix elements:

$$\tilde{s}^{(p)} = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(-j\beta_{m,-}^{(p)}\Delta y_p\right) \end{bmatrix} s^{(p)} \begin{bmatrix} \exp\left(j\beta_{m,+}^{(p)}\Delta y_p\right) & 0 \\ 0 & 1 \end{bmatrix}$$

# S-Matrices: Transverse Periodic Solution

- Divide into layers, uniform in z-direction
- Find Bloch states in each layer
- Calculate transfer function for field amplitudes
- Iteratively develop S-matrix
- Choose inputs from both sides
- Calculate resulting outputs (transmission and reflection) and losses (absorption $A=1-T-R$)

Whittaker & Culshaw, *Phys. Rev. B* **60**, 2610 (1999)
Tikhodeev *et al.*, *Phys. Rev. B* **66**, 045102 (2002)

# S-Matrices: Periodic Solution Strategy

- Can use a Fourier series expansion in real space of the H-fields:

$$\mathbf{H}(\mathbf{r},z)=\sum_{\mathbf{G}}\left(\phi_x(\mathbf{G})\left[\hat{\mathbf{x}}-\frac{1}{q}(k_x+G_x)\hat{\mathbf{z}}\right]+\phi_y(\mathbf{G})\left[\hat{\mathbf{y}}-\frac{1}{q}(k_y+G_y)\hat{\mathbf{z}}\right]\right)e^{i(\mathbf{k}+\mathbf{G})\cdot\mathbf{r}+iqz}$$

- In momentum space, can represent as:

$$h(z)=e^{iqz}\left\{\phi_x\hat{\mathbf{x}}+\phi_y\hat{\mathbf{y}}-\frac{1}{q}(\hat{k}_x\phi_x+\hat{k}_y\phi_y)\hat{\mathbf{z}}\right\}$$

- And then the electric field as:

$$\begin{aligned}e(z)=\frac{1}{q}e^{iqz}\,\hat{\eta}\{&[\hat{k}_y\hat{k}_x\phi_x+(q^2+\hat{k}_y\hat{k}_y)\phi_y]\hat{\mathbf{x}}\\&-[\hat{k}_x\hat{k}_y\phi_y+(q^2+\hat{k}_x\hat{k}_x)\phi_x]\hat{\mathbf{y}}+q[\hat{k}_y\phi_x-\hat{k}_x\phi_y]\hat{\mathbf{z}}\}\end{aligned}$$

# S-Matrices: Periodic Solution Strategy

- Eigenvalue equation becomes

$$\left\{\begin{pmatrix} \hat{\eta} & 0 \\ 0 & \hat{\eta} \end{pmatrix}\left[q^2+\begin{pmatrix} \hat{k}_x\hat{k}_x & \hat{k}_x\hat{k}_y \\ \hat{k}_y\hat{k}_x & \hat{k}_y\hat{k}_y \end{pmatrix}\right]+\begin{pmatrix} \hat{k}_y\hat{\eta}\hat{k}_y & -\hat{k}_y\hat{\eta}\hat{k}_x \\ -\hat{k}_x\hat{\eta}\hat{k}_y & \hat{k}_x\hat{\eta}\hat{k}_x \end{pmatrix}\right\}\times\begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix}=\omega^2\begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix}$$

- More compactly represented as:

$$[\mathcal{H}(q^2+K)+\mathcal{K}]\phi=\omega^2\phi$$

- Where the eigenvectors $\phi$ have a unique orthonormality condition:

$$\phi_n^T(\omega^2-\mathcal{K})\phi_{n'}=\delta_{nn'}$$

# S-Matrix Method: Advantages

- No *ad hoc* assumptions regarding structures
- Applicable to wide variety of problems
- Suitable for eigenmodes or high-$Q$ resonant modes at single frequency
- Can treat layers with large difference in length scales
- Computationally tractable enough on single core machines

# S-Matrix Method: Disadvantages

- Accurate solutions obtained more slowly as the following increase:
  - Number of layers
  - Absolute magnitude of Fourier components (especially for metals)
  - Number of plane-wave components ($\sim N^3$)
- Relatively slow for broad-band problems (time-domain is a good alternative)

# Next Class

- Is Wednesday, Mar. 1
- Next time, we will continue with transfer matrix models