ECE 695
Numerical Simulations
Lecture 26: Applying Coupled Mode Theory

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Coupled Mode Theory: Band Pass Filtering

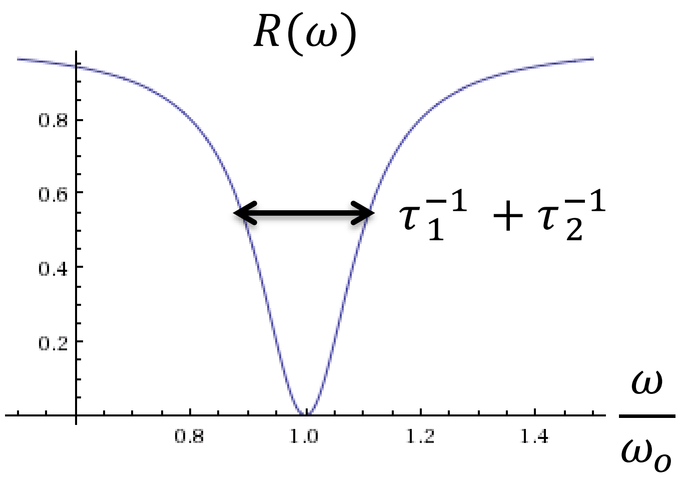
- For simple case: 2 waveguides + 1 resonator with 1 input:

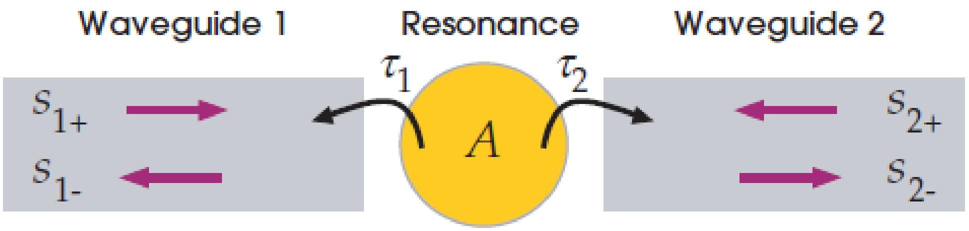
$$S_{1-} = S_{1+} - \sqrt{2/\tau_1}A$$
$$S_{2-} = \sqrt{2/\tau_2}A$$
$$\frac{dA}{dt} = -j\omega_o A - \frac{A}{\tau_1} - \frac{A}{\tau_2} + \sqrt{\frac{2}{\tau_1}}S_{1+}$$

- Reflection can be calculated as quotient:

$$R(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \frac{(\omega-\omega_o)^2 + (\tau_1^{-1} - \tau_2^{-1})^2}{(\omega-\omega_o)^2 + (\tau_1^{-1} + \tau_2^{-1})^2}$$

- Result: a Lorentzian dip in transmission, centered at resonant frequency ω_o





H. Haus, *Waves & Fields in Optoelectronics*, Chap. 7 (1984)
W. Suh et al., *IEEE J. Quantum Electron.* **40**, 1511 (2004)
J.D. Joannopoulos et al., *Photonic Crystals*, Chap. 10 (2008).

Numerical ODE Solvers

- Objective: to solve ODE (e.g., $\frac{d\boldsymbol{X}}{dt} = \boldsymbol{f}(\boldsymbol{X})$) with greatest accuracy and least computational cost
- Categories:
  - Initial value problems
  - Boundary value problems
- Algorithms:
  - Euler methods
  - Higher-order methods (e.g., Runge-Kutta)
  - Shooting methods
  - Finite element/difference methods

Numerical ODE IVP Solvers

- Euler Method: discretizes original ODE and solves in time steps of Δt:

$$\Delta \boldsymbol{X} = \Delta t \cdot \boldsymbol{f}(\boldsymbol{X})$$

- Advantages: fast, easy to implement
- Disadvantages: inaccurate for many ODEs with modest to large step sizes
- Problem is implicit linear evolution – fails badly for ODEs with significant curvature

Numerical ODE IVP Solvers

- Higher-order methods: incorporate more than just first derivative into solution
- In principle, incorporating nth derivative should reduce errors to $1/\Delta t^{n+1}$
- Example: leapfrog method
  - Let:

$$\frac{d}{dt}\binom{x}{v} = \binom{v}{F(x)}$$

  - Alternate updating x and v:

$$x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} F(x_i) \Delta t^2$$

$$v_{i+1} = v_i + \frac{1}{2}[F(x_i) + F(x_{i+1})] \Delta t$$

Numerical ODE IVP Solvers

- Runge-Kutta methods: incorporate points with spacing less than Δt for each higher accuracy (e.g., ode45 in MATLAB)
- Generally define k_i to incrementally determine slopes within interval:

$$k_i = \Delta t \cdot f(t_n + a_i \Delta t, y_n + \sum_{j=1} b_{ij} k_j)$$

- This gives rise to solutions:

$$y_{n+1} = y_n + \Delta t \sum_{i=1}^{M} c_i k_i$$

- In general, order of solution will be equal to *M*; accuracy will go as $1/\Delta t^{M+1}$

Numerical ODE Solvers

- Stiff solvers: for ODEs with a rapid harmonic oscillation, use backward differentiation formulae:

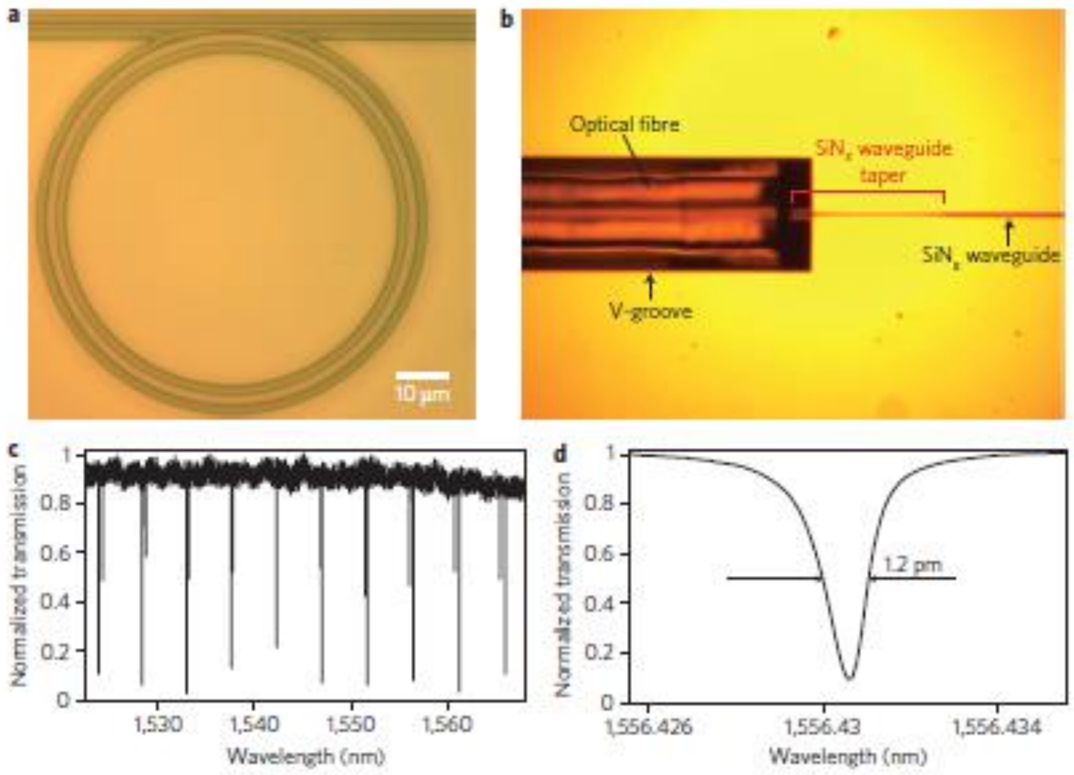
$$\sum_{i=0}^{M} c_i y_{n+i} = \Delta t \cdot f(t_{n+M}, y_{n+M})$$

- Convergence can be orders of magnitude better
- Implemented with ode15s in MATLAB

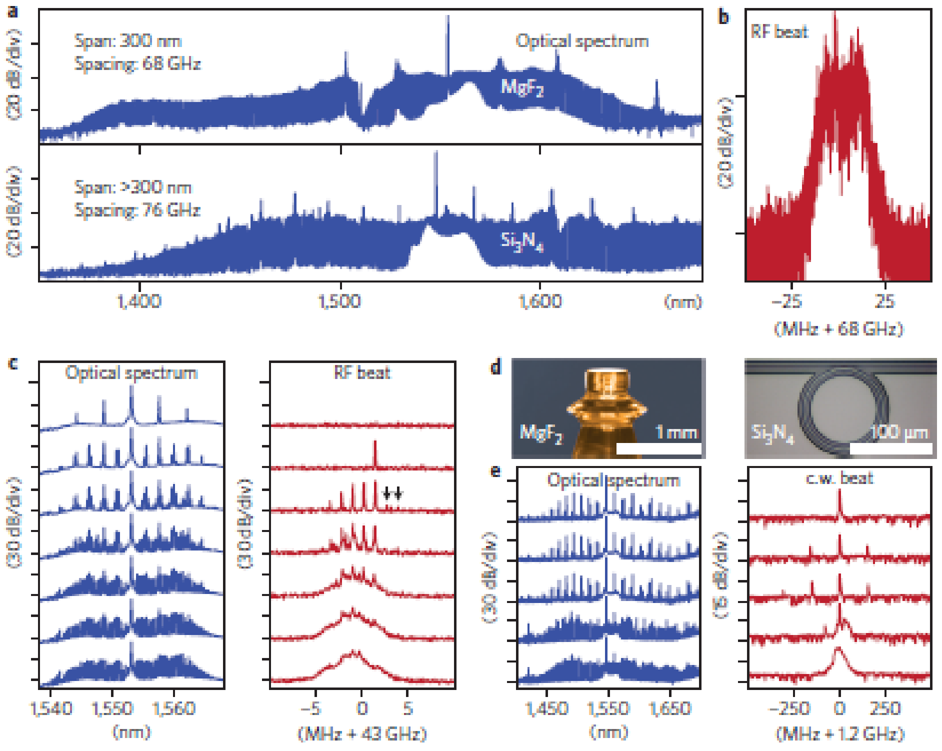
Numerical ODE BVP Solvers

- Shooting method applies Euler method
- However, finite element and finite difference methods are ideal for boundary value problems
- Finite elements: discretize on finite element basis, and solve using Galerkin method
- Finite difference: discretize on grid, and solve using leapfrog method

Microring combs

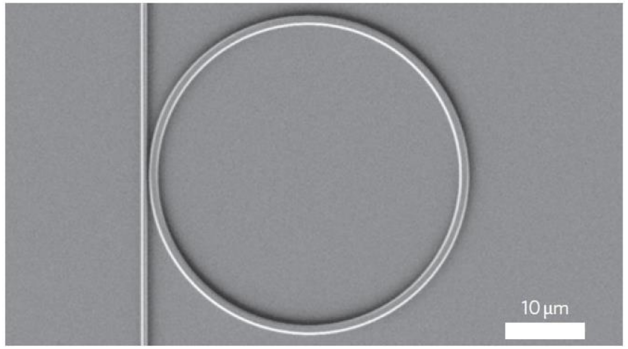


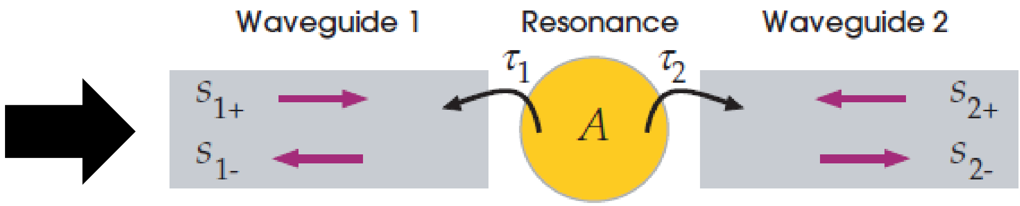
Ferdous, Fahmida, et al. "Spectral line-by-line pulse shaping of on-chip microresonator frequency combs." *Nature Photonics* 5.12 (2011): 770-776.



Herr, T., et al. "Universal formation dynamics and noise of Kerr-frequency combs in microresonators." *Nature Photonics* 6.7 (2012): 480-487.

CMTComb3 – a nanoHUB.org tool

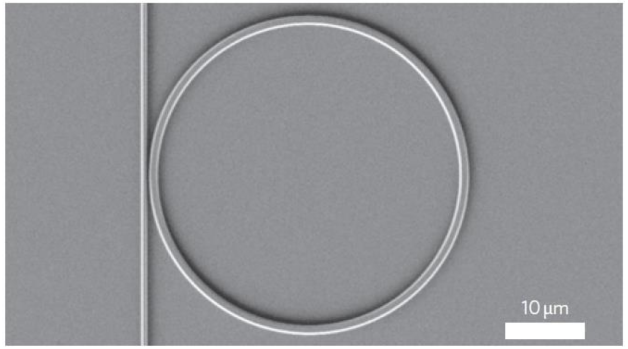


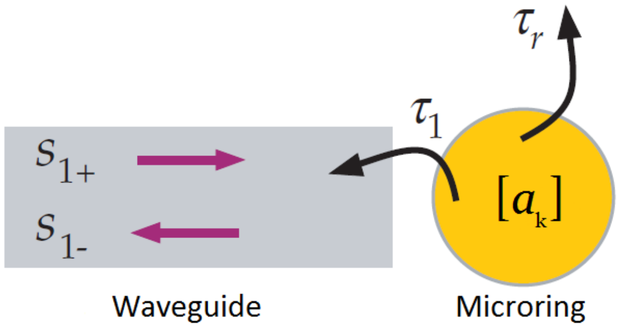


J.D. Joannopoulos et al., *Photonic Crystals*, Chap. 10 (2008).

Goal is to reduce physics of four-wave mixing in a microring resonator to a set of coupled mode equations, then solve them

Advantages of CMT/FDTD approach





- Vastly reduce complexity: 10^7 voxels → 10^2 modes
- Provides insight into system dynamics
- Allows full exploration of possible behaviors

CMTComb3 – a nanoHUB.org tool

- Coupled mode theory (CMT) is accurate in the weak-coupling regime, and much faster than full-wave time-domain simulations of comparable systems

- This CMT comb simulation enables exploration of basic physics and rapid prototyping for specific applications, including metrology and RF signal modulation

CMTComb3 – a nanoHUB.org tool

- Basic equations

Loss terms

Nonlinear coupling terms

Pump input

Pump mode: $$\frac{da_k}{dt} = (i\omega_k - \Gamma_k - \gamma_k)a_k + i\sum_{l,m}^{N} k_{lmk}a_l{}^* a_m a_{k+l-m} + \sqrt{2\Gamma_k}s_+$$

Side modes: $$\frac{da_k}{dt} = (i\omega_k - \Gamma_k - \gamma_k)a_k + i\sum_{l,m}^{N} k_{lmk}a_l{}^* a_m a_{k+l-m}$$

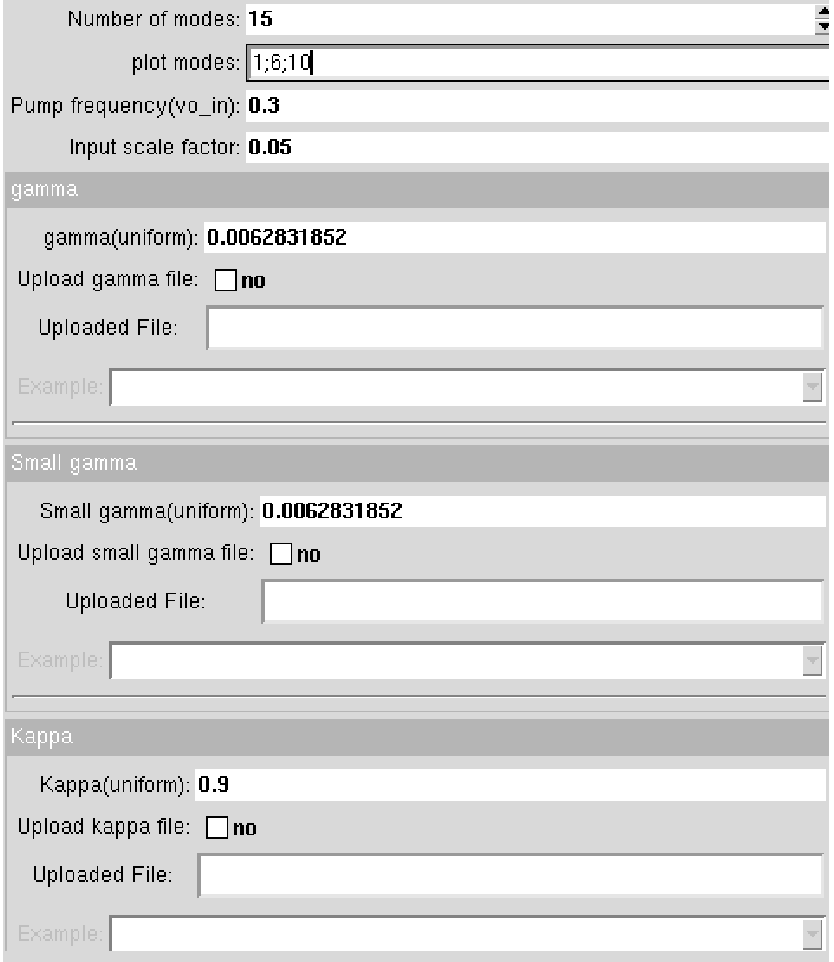
Output mode: $$\frac{ds_-}{dt} = -\frac{ds_+}{dt} + \sum_{k=1}^{N}\sqrt{2\Gamma_k}\frac{da_k}{dt}$$

R. Shugayev *et al.*, "Coupled Mode Theory for Microring Resonator Cavities," presented at *International Workshop for Novel Ideas in Optics* (June 2012, West Lafayette, IN).

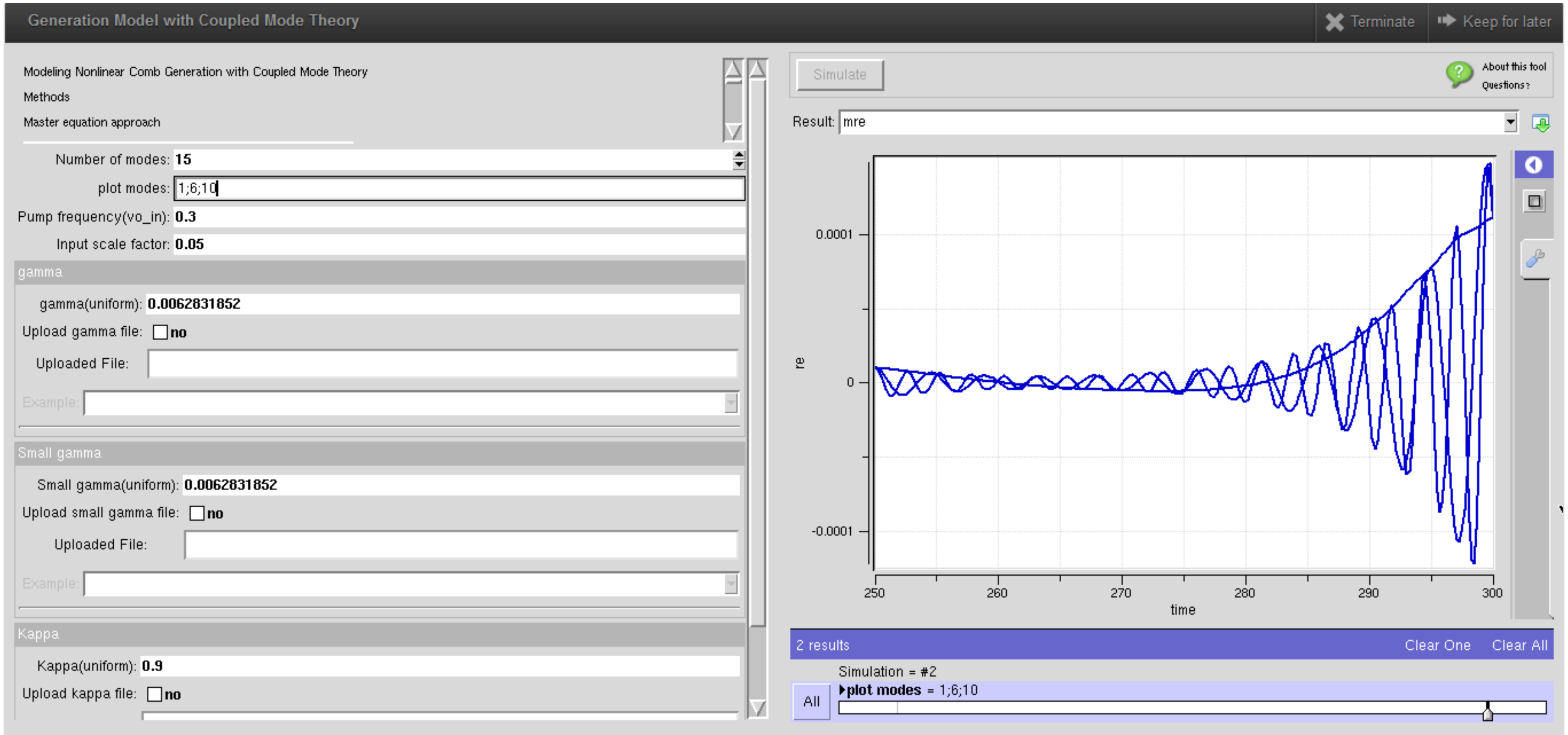
CMTComb3 – a nanoHUB.org tool

Input parameters for user to modify:

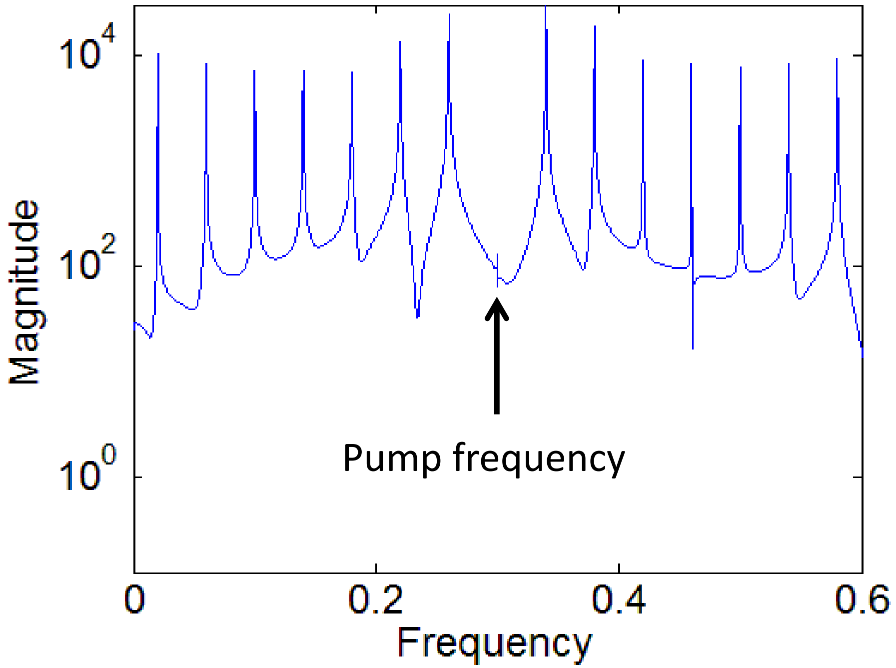
- Number of modes
- Plot modes
- Pump frequency
- Input scale factor
- Loss rate Γ
- Nonlinear strength κ



CMTComb3 – a nanoHUB.org tool



Pump filter results



Output spectrum of the system with the added filter

- Pump frequency level can be arbitrarily defined with appropriate attenuation in the branch
- Coupling is assumed the same as for the cavity modes

Cavity mode profile.

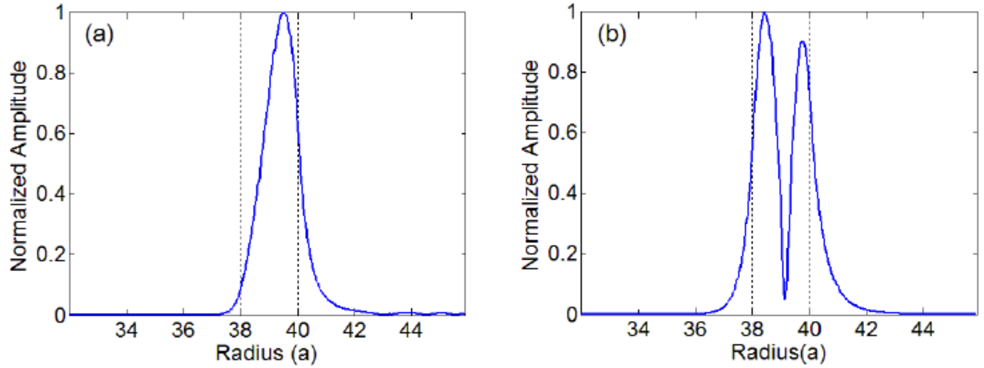


Fig. 1. (a) Radial distribution of the normalized amplitude function $|g(r)|$ for the fundamental normal dispersion mode (b) Radial distribution for the order 1 anomalous dispersion mode. The boundaries of the microring core region are shown as dotted lines.

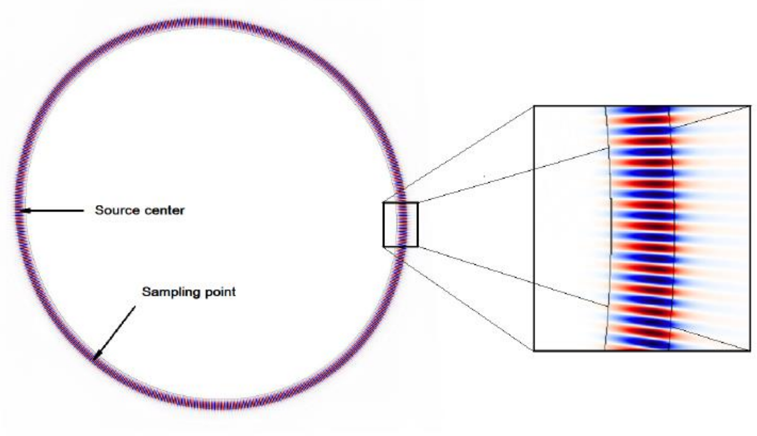


Fig. 2. Distribution of the electric field of a mode with a fundamental radial profile. The component of the field going into the plane (E_z) is shown. Red designates positive values; blue negative; white zero.

- Normally Bessel. Often nonanalytic: Racetrack, coupled microring/waveguide, Microring/microring
- Predominantly single-mode structures can be suitably excited
- Gaussian mode profile
- Quasi 2D simulated modes . Can be extended to 3D

Dispersion implementation. FDTD

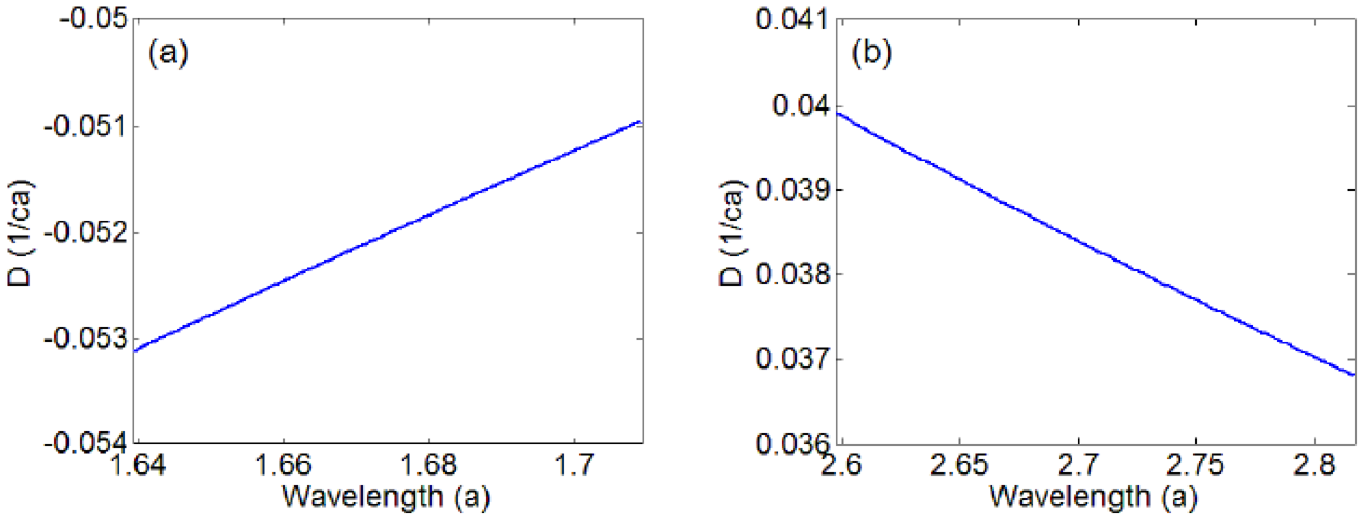


Fig. 3. (a) Dispersion parameter data for normal dispersion region of fundamental radial (order 0) profile modes; (b) the anomalous dispersion region of order 1 modes.

- Material dispersion of different sign can be introduced – too lossy for reasonably-timed simulations
- Geometric dispersion of different family modes allows to test anomalous and normal dispersion behavior

Comparison of normal to anomalous dispersion

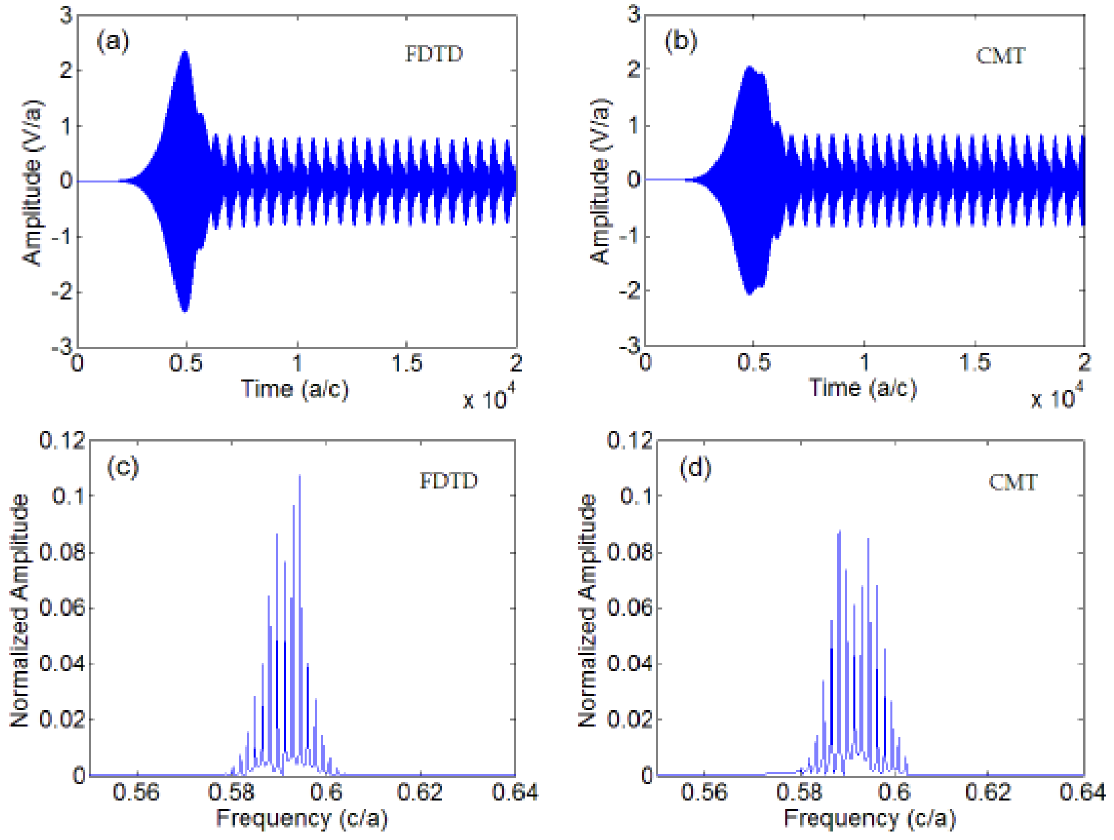


Fig. 4. Comparison of FDTD and CMT results for the normal dispersion case: (a) FDTD time domain; (b) CMT time domain; (c) FDTD frequency domain; and (d) CMT frequency domain.

- Normal dispersion – good match in the time domain.
- Good qualitative match in the frequency domain (bandwidth, power)

Separation of different interaction terms

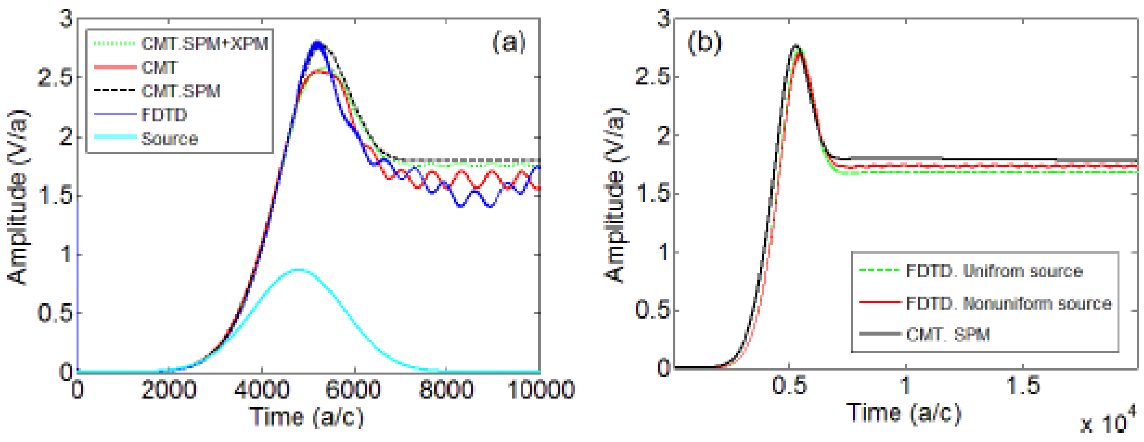


Fig. 6. Time domain envelopes of different coupling and excitation regimes in FDTD and CMT: (a) Comparison of FDTD and CMT envelopes for different coupling mechanisms: (solid blue) FDTD results, (solid red) CMT results, (dashed black) CMT results with only self-phase modulation (SPM) terms, (dotted green) CMT results with only SPM and cross-phase modulation (XPM) terms, (solid cyan) amplitude scaled source transient; (b) Comparison of FDTD and CMT for different excitation regimes: (dotted green) uniform source, (solid red) source with $\sigma = 3.14$, (solid black) CMT results with only SPM terms and $\sigma = 0.1$.

- Mismatch is attributed to nonuniform source excitation profile
- Model assumes uniformity of the coupling –nonuniform coupling can be incorporated
- Should not be an issue with soft excitation

Soliton pulses continued

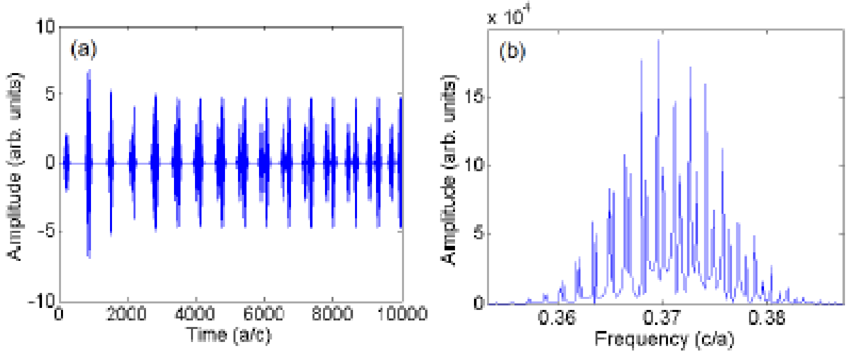


Fig. 8. (a) Time domain traveling pulses in the "cold" microring cavity with an order 1 source profile. (b) Corresponding frequency domain data, showing that the broadband excitation spectrum excites both order 0 and order 1 modes.

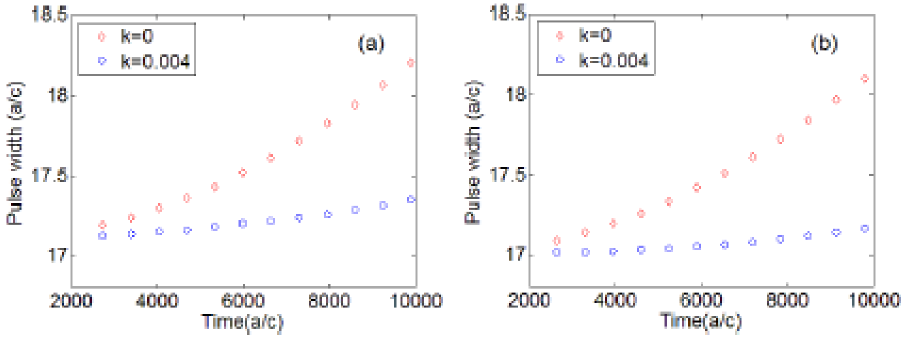


Fig. 9. Evolution of the propagating pulse in the cases of zero and $X^{(3)} = 0.004$ Kerr nonlinearity: (a) FDTD calculation (b) corresponding CMT simulation. A good match is observed. In both cases, pulse spreading is dramatically suppressed with appropriate level of nonlinearity, much like in a soliton.

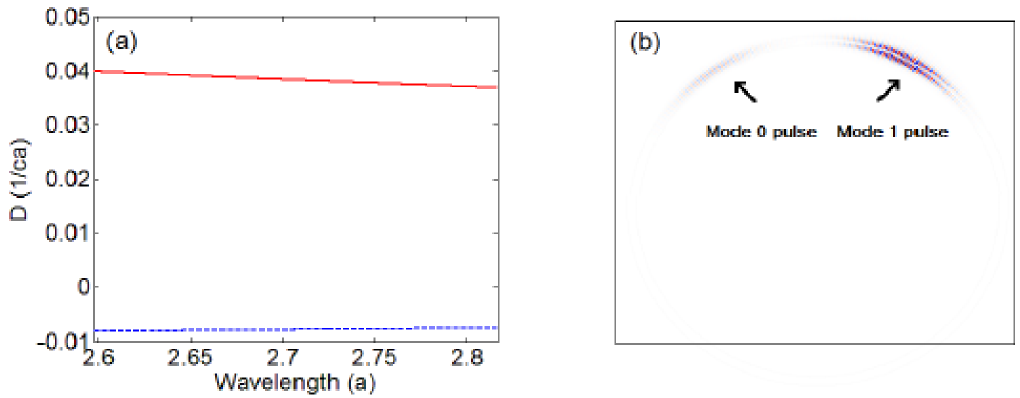
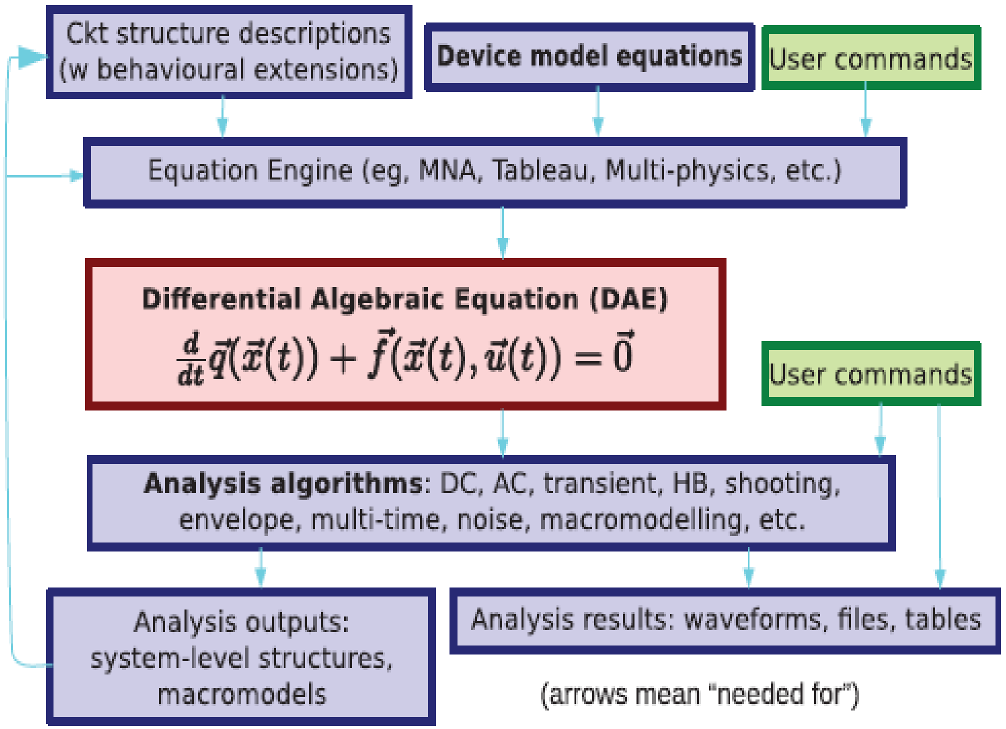


Fig. 7. (a) Dispersion parameter data for order 0 modes (dotted blue) having normal dispersion and order 1 modes (solid red) with anomalous dispersion; (b) Field profile of the traveling pulses corresponding to order 0 modes and order 1 modes. All simulations were performed in the "cold" cavity.

- Modes corresponding to different modal families have significantly different group velocities- hence low frequency periodicity in coupling of interfamily modes close in frequency.

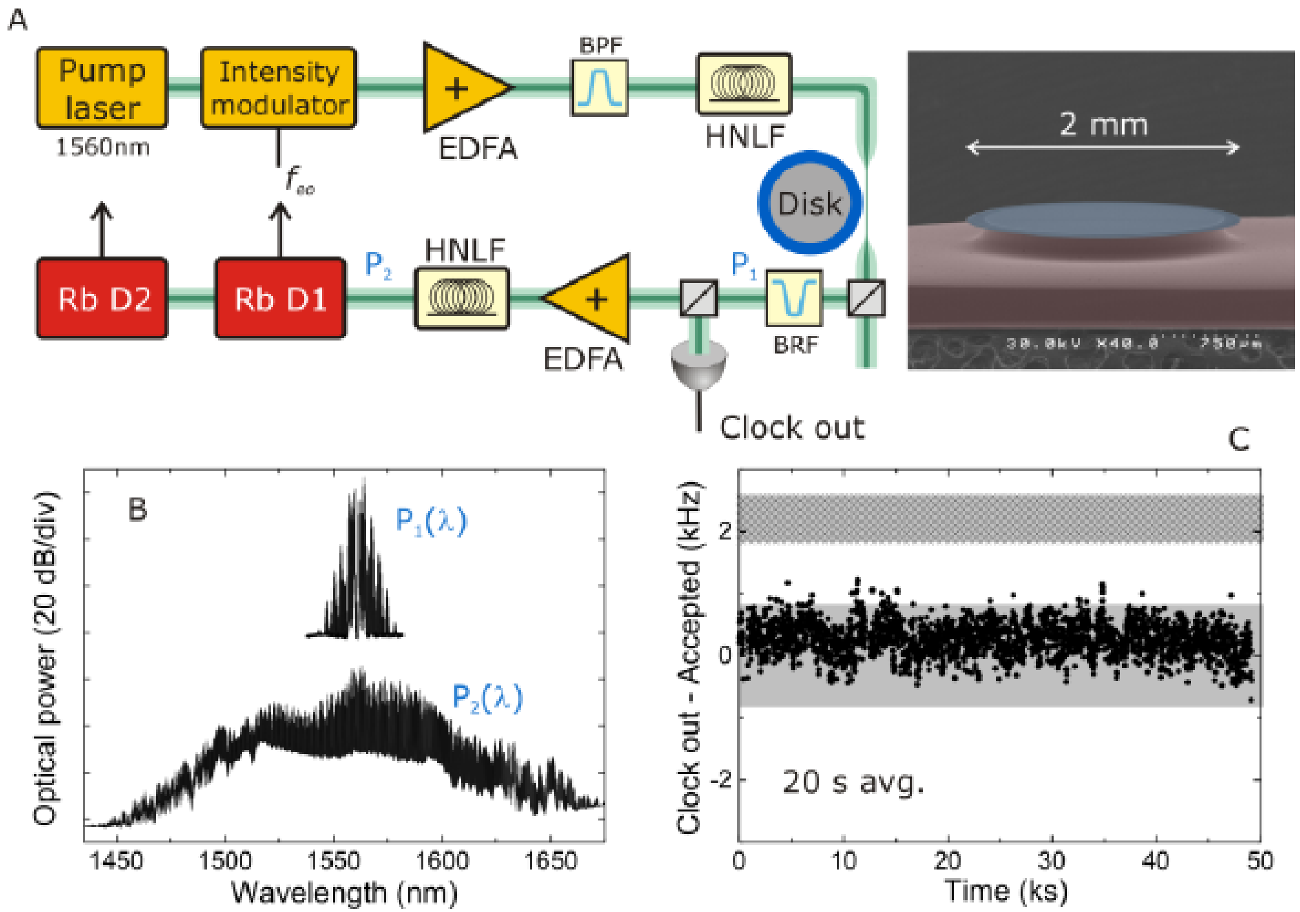
Using MAPP to generate VHDL models of microring resonators



- MAPP permits definition and higher
  level interconnects
- MAPP optical module
- High level abstraction

Wang, Tianshi, et al. "MAPP: the berkeley model and algorithm prototyping platform." Custom Integrated Circuits Conference (CICC), 2015 IEEE. IEEE, 2015

Frequency comb optical clocks



- The clock signal is produced by detecting the beat frequency with the photodetector
- Single FSR combs generated using parametric seeding – intensity modulation of the pump input

Papp, Scott B., et al. "Microresonator frequency comb optical clock." *Optica*1.1 (2014): 10-14.

Conclusions

- In general, CMT works for a broad range of systems with well-defined and relatively weakly coupled resonances
- Can be readily extended to cases with weak losses, by treating them as additional 'waveguides'
- Furthermore, in the linear case, most problems can be solved analytically
- Can extend CMT to nonlinear systems (e.g., Kerr media) or time-varying systems, but generally must use ODE solvers to find numerical solutions
- Can construct one's own solver in MATLAB, or use CMTcomb3 on nanoHUB

Next Class

- Next time: we will discuss finite-difference time domain techniques
- Suggested reference: S. Obayya's book, Chapter 5, Sections 4-6