In the previous lesson we showed how to solve quadratic equations that were not factorable and were not perfect squares by making perfect square trinomials using a process called completing the square. In this lesson we will see another method for solving quadratic equations which are not factorable and are not perfect squares by using a formula called **<u>the quadratic formula</u>**, which is derived from completing the square.

## <u>Quadratic Formula:</u>

- another method for solving quadratic equations ($ax^2 + bx + c = 0$)
  - $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- the result of completing the square for the standard quadratic equation $ax^2 + bx + c = 0$
  - if you took $ax^2 + bx + c = 0$ and followed the steps for completing the square, you would derive the quadratic formula
- unlike when solving quadratic equations by completing the square, you do **<u>NOT</u>** have to make the leading coefficient of the quadratic equation one when using the quadratic formula
- if you want to eliminate fractions or decimals, you may, but it is not required:
  - $-0.6x^2 + 1.3x - 1 = 0$
  - $\frac{2}{3}x^2 - \frac{1}{6}x - 4 = 0$

$a = -0.6\ ;\ b = 1.3\ ;\ c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(1.3) \pm \sqrt{(1.3)^2 - 4(-0.6)(-1)}}{2(-0.6)}$$

$\vdots$

$a = \frac{2}{3}\ ;\ b = -\frac{1}{6}\ ;\ c = -4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-\left(-\frac{1}{6}\right) \pm \sqrt{\left(-\frac{1}{6}\right)^2 - 4\left(\frac{2}{3}\right)(-4)}}{2\left(\frac{2}{3}\right)}$$

$\vdots$

**<u>Example 1:</u>** Solve the following equation for $x$ and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there is no real solution, enter NO SOLUTION.

$$x^2 + 3x + 1 = 0$$

$$a = 1 \quad ; \quad b = 3 \quad ; \quad c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## Steps for Solving Quadratic Equations using the Quadratic Formula:

1. write the equation in polynomial form and it set equal to zero
   - $ax^2 + bx + c = 0$
2. identify the coefficients $a, b$, and $c$, and plug them into the formula
   $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3. **SIMPLIFY COMPLETELY**
   - simplify your square root completely
     - look for perfect squares $(4, 9, 16, \ldots)$, or for numbers that contain factors that are perfect squares $(8, 45, 96, \ldots)$
   - simplify the fraction completely
     - factor the numerator and cancel common factors with the denominator, or write two separate fractions and simplify each

Plugging the quadratic equation $7x^2 - 2x - 7 = 0$ into the quadratic formula results in the following:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(7)(-7)}}{2(7)}$$

$$x = \frac{2 \pm \sqrt{4 + 196}}{14}$$

$$x = \frac{2 \pm \sqrt{200}}{14}$$

$$x = \frac{2 \pm \sqrt{100}\sqrt{2}}{14}$$

$$x = \frac{2 \pm 10\sqrt{2}}{14}$$

$$x = \frac{2}{14} \pm \frac{10\sqrt{2}}{14}$$

$$x = \frac{1}{7} \pm \frac{5\sqrt{2}}{7}$$

$$\boldsymbol{x = \frac{1}{7} + \frac{5\sqrt{2}}{7} \quad , \quad \frac{1}{7} - \frac{5\sqrt{2}}{7}}$$

200 is not a perfect square, but it contains a factor of 100 which is, so that is why I broke $\sqrt{200}$ into $\sqrt{100}\sqrt{2}$.

To simplify the fraction, you could factor out a 2 from the numerator and then cancel with the denominator, or you can break the fraction $\frac{2 \pm 10\sqrt{2}}{14}$ into two separate fractions, $\frac{2}{14}$ and $\frac{10\sqrt{2}}{14}$ in order to simplify each individually. Either way is acceptable.

**<u>Example 2:</u>** Solve the following equations for $x$ and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a. $x^2 + 10x + 12 = 0$

$a = 1$ ; $b = 10$ ; $c = 12$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(10) \pm \sqrt{(10)^2 - 4(1)(12)}}{2(1)}$

$x = \frac{-10 \pm \sqrt{52}}{2}$

$x = \frac{-10 \pm \sqrt{4}\sqrt{13}}{2}$

$x = -\frac{10}{2} \pm \frac{2\sqrt{13}}{2}$

$x = -5 \pm \sqrt{13}$

$\boldsymbol{x = -5 + \sqrt{13}\ , -5 - \sqrt{13}}$

b. $3x^2 - 3x - 4 = 0$

$a = 3$ ; $b = -3$ ; $c = -4$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-4)}}{2(3)}$

c. $3x^2 = 2x - 1$

$3x^2 - 2x + 1 = 0$

$a = 3 \ ; \ b = -2 \ ; \ c = 1$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(1)}}{2(3)}$

$x = \frac{2 \pm \sqrt{4 - 12}}{6}$

$x = \frac{2 \pm \sqrt{-8}}{6}$

Since the square root of $-8$ (or the square root of negative anything) does not exist with real numbers, the equation $3x^2 = 2x - 1$ has **NO SOLUTION**.

d. $\frac{7}{x^2} - \frac{10}{x} - 1 = 0$

$x^2 \cdot \left(\frac{7}{x^2} - \frac{10}{x} - 1\right) = (0) \cdot x^2$

$7 - 10x - x^2 = 0$

$a = -1 \ ; \ b = -10 \ ; \ c = 7$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(-1)(7)}}{2(-1)}$

Keep in mind that even if you do everything correctly when solving a quadratic equation using the quadratic formula, you are not guaranteed to get real solutions. If a quadratic equation has no real solutions, that will be revealed regardless of how you solve the equation (completing the square, quadratic formula, etc.).

**Example 3:** Solve the following equations for $x$ and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

a.

$$\frac{2x}{x-3}+\frac{5}{x+3}=\frac{36}{x^2-9}$$

$$(x-3)(x+3)\left(\frac{2x}{x-3}+\frac{5}{x+3}\right)=\left(\frac{36}{x^2-9}\right)(x-3)(x+3)$$

$$2x(x+3)+5(x-3)=36$$

$$2x^2+6x+5x-15=36$$

$$2x^2+11x-15=36$$

$$2x^2+11x-51=0$$

| ***ac*** | ***b*** | **Think about the signs of the product and the sum.** |
|---|---|---|
| | | |
| | | |
| | | |
| | | |

b.

$$1=\frac{5}{x}-\frac{20}{x^2}$$

$$x^2(1)=\left(\frac{5}{x}-\frac{20}{x^2}\right)x^2$$

$$x^2=5x-20$$

$$x^2-5x+20=0$$

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(20)}}{2(1)}$$

$$x=\frac{5\pm\sqrt{25-80}}{2}$$

$$x=\frac{5\pm\sqrt{-55}}{2}$$

Since the square root of $-55$ (or negative anything) does not exist with real numbers, the equation $1=\frac{5}{x}-\frac{20}{x^2}$ has **NO SOLUTION**.

c.

$$\frac{1}{x}+\frac{1}{x+2}=\frac{1}{4}$$

$$4x(x+2)\left(\frac{1}{x}+\frac{1}{x+2}\right)=\left(\frac{1}{4}\right)4x(x+2)$$

$$4(x+2)+4x=x(x+2)$$

$$4x+8+4x=x^2+2x$$

$$8x+8=x^2+2x$$

$$0=x^2-6x-8$$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-8)}}{2(1)}$$

$$x=\frac{6\pm\sqrt{36+32}}{2}$$

$$x=\frac{6\pm\sqrt{68}}{2}$$

$$x=\frac{6\pm\sqrt{4}\sqrt{17}}{2}$$

$$x=\frac{6\pm2\sqrt{17}}{2}$$

$$x=\frac{6}{2}+\frac{2\sqrt{17}}{2}\ ,\ \frac{6}{2}-\frac{2\sqrt{17}}{2}$$

$$\boldsymbol{x=3+\sqrt{17}\ ,\ 3-\sqrt{17}}$$

d.

$$\frac{3}{x-3}+\frac{5}{x-4}=\frac{x^2-19}{x^2-7x+12}$$

$$(x-3)(x-4)\left(\frac{3}{x-3}+\frac{5}{x-4}\right)=\left(\frac{x^2-19}{x^2-7x+12}\right)(x-3)(x-4)$$

$$3(x-4)+5(x-3)=x^2-19$$

$$3x-12+5x-15=x^2-19$$

$$8x-27=x^2-19$$

$$0=x^2-8x+8$$

e.

$$\frac{2x}{x-3}+\frac{6}{x+3}=-\frac{28}{x^2-9}$$

$(x-3)(x+3)\left(\frac{2x}{x-3}+\frac{6}{x+3}\right)=\left(-\frac{28}{x^2-9}\right)(x-3)(x+3)$

$2x(x+3)+6(x-3)=-28$

$2x^2+6x+6x-18=-28$

$2x^2+12x-18=-28$

$2x^2+12x+10=0$

f.

$$\frac{-2x}{x^2+6x+8}=\frac{x}{x+4}-\frac{2}{x+2}$$

$(x+2)(x+4)\left(\frac{-2x}{x^2+6x+8}\right)=\left(\frac{x}{x+4}-\frac{2}{x+2}\right)(x+4)(x+2)$

$-2x=x(x+2)-2(x+4)$

$-2x=x^2+2x-2x-8$

$-2x=x^2-8$

$0=x^2+2x-8$

$x=\frac{-(2)\pm\sqrt{(2)^2-4(1)(-8)}}{2(1)}$

$x=\frac{-2\pm\sqrt{4+32}}{2}$

$x=\frac{-2\pm\sqrt{36}}{2}$

$x=\frac{-2\pm6}{2}$

$x=\frac{-2+6}{2}\ ,\ \frac{-2-6}{2}$

$x=2\ ,-4$

$x=-4$ is not a valid answer because it would make the original equation have denominators of zero. So the only valid answer is $\boldsymbol{x=2}$.

Every quadratic equation can be solved by either completing the square or by using the quadratic formula. While many students prefer the quadratic formula, keep in mind that the quadratic formula is limited to solving only quadratic equations, while completing the square can be used to solve other non-quadratic equations (as we will see in Lesson 15).

*Answers to Exercises:*

*1.* $x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$ ; *2a.* $x = -5-\sqrt{13}, -5+\sqrt{13}$ ;

*2b.* $x = \frac{3-\sqrt{57}}{6}, \frac{3+\sqrt{57}}{6}$; *2c.* $NO\ SOLUTION$ ;

*2d.* $x = -5-4\sqrt{2}, -5+4\sqrt{2}$; *3a.* $x = -\frac{17}{2}$ ; *3b.* $NO\ SOLUTION$ ;

*3c.* $x = 3+\sqrt{17}, 3-\sqrt{17}$ ; *3d.* $x = 4-2\sqrt{2}, 4+2\sqrt{2}$;

*3e.* $x = -5, -1$ ; *3f.* $x = 2$ ;