In Lessons 12, 13, and 14 we have seen that quadratic equations are equations that can be written in the form $ax^2 + bx + c = 0$. In this lesson we will see that certain polynomial equations which are not quadratic can still be solved using some of the same methods as quadratics. For instance, the equation $x^4 - 2x^2 - 8 = 0$ is not quadratic because the degree of the polynomial is 4 rather than 2; however, it can still be solved by factoring, just like a quadratic equation.

Steps for Solving an Equation by Factoring:

1. write the equation as a polynomial and set it equal to zero
2. factor the polynomial
3. use Zero Factor Theorem to solve

Example 1: Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

ac **-8**	***b*** **-2**
$-1, 8$	7
$1, -8$	-7
$-2, 4$	2
$\mathbf{2, -4}$	$\mathbf{-2}$

$$x^4 - 2x^2 - 8 = 0$$

$$x^4 + 2x^2 - 4x^2 - 8 = 0$$

$$x^2(x^2 + 2) - 4(x^2 + 2) = 0$$

$$(x^2 + 2)(x^2 - 4) = 0$$

$$x^2 + 2 = 0 \quad ; \quad x^2 - 4 = 0$$

$$x^2 = -2 \quad ; \quad x^2 = 4$$

x^2 cannot be equal to -2 using real numbers

Remember that a perfect square (such as x^2) cannot be equal to a negative value when you are working with real numbers. So when you have $x^2 = -2$, you should recognize that this is not possible with real numbers and thus we eliminate that equation. If you go ahead and take the square root of both sides of the equation, you would end up with $= \pm\sqrt{-2}$. Once again this is not possible with real numbers, so we eliminate that equation.

$$x^2 = 4$$

$$x = \pm 2$$

$$\boldsymbol{x = -2, 2}$$

<u>Always be sure to check that the value that is equal to a perfect square, or under a square root, is non-negative. Also, keep in mind that when taking the square root of both sides of an equation, we have a positive root and a negative root.</u>

<u>Example 2:</u> Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

Any radicals and/or fractions in your final answer should be simplified completely.

a. $6x^4 + 29x^2 + 28 = 0$

ac 168	***b*** 29	**Think about the signs of the product and the sum.**
1, 168	69	
2, 84	86	
3,56	59	
4,42	46	
6,28	34	
7,24	31	
8, 21	29	

b. $75x^4 + 65x^3 - 10x^2 = 0$

$$5x^2(15x^2 + 13x - 2) = 0$$

$$5x^2 = 0 \quad ; \quad 15x^2 + 13x - 2 = 0$$

$$x^2 = 0 \quad ; \quad 15x^2 - 2x + 15x - 2 = 0$$

$$x = \pm\sqrt{0} \quad ; \quad x(15x - 2) + 1(15x - 2) = 0$$

$$x = 0 \quad ; \quad (15x - 2)(x + 1) = 0$$

$$x = 0 \quad ; \quad 15x - 2 = 0 \quad ; \quad x + 1 = 0$$

$$x = 0 \quad ; \quad 15x = 2 \quad ; \quad x = -1$$

$$x = 0 \quad ; \quad x = \frac{2}{15} \quad ; \quad x = -1$$

$$\boldsymbol{x = 0, \frac{2}{15}, -1}$$

c. $x^4 - 15x^2 - 54 = 0$

$\underline{ac}$ -54	$\underline{b}$ -15
$1, -54$	-53
$2, -27$	-25
$\mathbf{3, -18}$	$\mathbf{-15}$

$x^4 + 3x^2 - 18x^2 - 54 = 0$

$x^2(x^2 + 3) - 18(x^2 + 3) = 0$

$(x^2 + 3)(x^2 - 18) = 0$

$x^2 + 3 = 0 \;\; ; \;\; x^2 - 18 = 0$

$x^2 = -3 \;\; ; \;\; x^2 = 18$

x^2 cannot be equal to -3

$x = \pm\sqrt{18}$

$\boldsymbol{x = 3\sqrt{2}, -3\sqrt{2}}$

d. $5x^4 - 13x^2 + 6 = 0$

$\underline{ac}$ 30	$\underline{b}$ -13
$-1, -30$	-31
$-2, -15$	-17
$\mathbf{-3, -10}$	$\mathbf{-13}$

$5x^4 - 3x^2 - 10x^2 + 6 = 0$

$x^2(5x^2 - 3) - 2(5x^2 - 3) = 0$

$(5x^2 - 3)(x^2 - 2) = 0$

$5x^2 - 3 = 0 \;\; ; \;\; x^2 - 2 = 0$

$5x^2 = 3 \;\; ; \;\; x^2 = 2$

$x^2 = \frac{3}{5} \;\; ; \;\; x = \pm\sqrt{2}$

$x = \pm\sqrt{\frac{3}{5}} \;\; ; \;\; x = \pm\sqrt{2}$

$\boldsymbol{x = \sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}, \sqrt{2}, -\sqrt{2}}$

We have already seen that certain polynomial equations which are not quadratic can be solved by factoring. However for an equation such as $x^4 - 2x^2 - 10 = 0$, which is not quadratic and not factorable, we can still solve it by completing the square.

<u>Steps for Completing the Square $\left(ax^{2n} + bx^n + c = 0\right)$</u>

1. divide each term by the leading coefficient a
2. isolate the constant term c
3. add $\left(\frac{b}{2}\right)^2$ to both sides
4. factor the polynomial as a perfect square trinomial
5. isolate the perfect square and solve by extracting square roots

<u>Example 3:</u> Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

Any radicals and/or fractions in your final answer should be simplified completely.

$\underline{\boldsymbol{ac}}$ $\mathbf{-10}$	$\underline{\boldsymbol{b}}$ $\mathbf{-2}$
$1, -10$	-9
$-1, 10$	9
$2, -5$	-3
$-2, 5$	3

$$x^4 - 2x^2 - 10 = 0$$

$$x^4 - 2x^2 + \_\_\_\_ = 10 + \_\_\_\_$$

$$x^4 - 2x^2 + \left(\frac{-2}{2}\right)^2 = 10 + \left(\frac{-2}{2}\right)^2$$

$$x^4 - 2x^2 + 1 = 10 + 1$$

$$(x^2 - 1)^2 = 11$$

$$x^2 - 1 = \pm\sqrt{11}$$

$$x^2 = 1 \pm \sqrt{11}$$

$$x^2 = 1 + \sqrt{11} \quad ; \quad x^2 = 1 - \sqrt{11}$$

x^2 cannot be equal to $1 - \sqrt{11}$ because $1 - \sqrt{11}$ is negative

Remember that a perfect square (such as x^2) cannot be equal to a negative value when you are working with real numbers. So when you have $x^2 = 1 - \sqrt{11}$, you should recognize that $1 - \sqrt{11}$ is a negative value. If you don't, and had simply taken the square root of both sides of the equation, you would end up with $= \pm\sqrt{1 - \sqrt{11}}$. At this point you would need to recognize that $\sqrt{1 - \sqrt{11}}$ does not exist because once again, $1 - \sqrt{11}$ is a negative value.

$$x^2 = 1 + \sqrt{11}$$

$$x = \pm\sqrt{1 + \sqrt{11}}$$

$$\boldsymbol{x = \sqrt{1 + \sqrt{11}}, -\sqrt{1 + \sqrt{11}}}$$

<u>Once again, ALWAYS be sure to check that the value that is equal to a perfect square, or under a square root, is non-negative.</u>

Final answers should be left in exact form unless the directions state otherwise. However, it might be helpful to approximate a final answer using a calculator when working with radicals to determine whether a radicand is positive or negative.

<u>Example 4:</u> Solve the following equations for x and enter exact answers only (no decimal approximations). If there is more than one solution, separate your answers with commas. If there are no real solutions, enter NO SOLUTION.

Any radicals and/or fractions in your final answer should be simplified completely.

a. $4x^4 + 12x^2 + 4 = 0$

$$\frac{4x^4}{4} + \frac{12x^2}{4} + \frac{4}{4} = \frac{0}{4}$$

$$x^4 + 3x^2 + 1 = 0$$

$$x^4 + 3x^2 + \left(\frac{3}{2}\right)^2 = -1 + \left(\frac{3}{2}\right)^2$$

$$x^4 + 3x^2 + \frac{9}{4} = -1 + \frac{9}{4}$$

$$\left(x^2 + \frac{3}{2}\right)^2 = \frac{5}{4}$$

$$x^2 + \frac{3}{2} = \pm\sqrt{\frac{5}{4}}$$

$$x^2 = -\frac{3}{2} + \sqrt{\frac{5}{4}} \quad ; \quad x^2 = -\frac{3}{2} - \sqrt{\frac{5}{4}}$$

$$-\frac{3}{2} + \sqrt{\frac{5}{4}} \approx -0.38 \ ; \ -\frac{3}{2} - \sqrt{\frac{5}{4}} \approx -2.62$$

Since both are negative, neither is possible.

NO SOLUTION

b. $3x^{-4} - 12x^{-2} + 1 = 0$

$$\frac{3}{x^4} - \frac{12}{x^2} + 1 = 0$$

$$x^4 \cdot \left(\frac{3}{x^4} - \frac{12}{x^2} + 1\right) = (0) \cdot x^4$$

$$3 - 12x^2 + x^4 = 0$$

$$x^4 - 12x^2 + 3 = 0$$

c. $x^4 - 2x^2 - 4 = 0$

$x^4 - 2x^2 + \left(\frac{-2}{2}\right)^2 = 4 + \left(\frac{-2}{2}\right)^2$

$x^4 - 2x^2 + 1 = 4 + 1$

$(x^2 - 1)^2 = 5$

$x^2 - 1 = \pm\sqrt{5}$

$x^2 = 1 \pm \sqrt{5}$

$x^2 = 1 + \sqrt{5}$; $x^2 = 1 - \sqrt{5}$

$1 - \sqrt{5} \approx -1.236$ *and* $1 - \sqrt{5} \approx -1.236$

x^2 cannot be equal to $1 - \sqrt{5}$

$x^2 = 1 + \sqrt{5}$

$x = \pm\sqrt{1 + \sqrt{5}}$

$\boldsymbol{x = \sqrt{1 + \sqrt{5}}, -\sqrt{1 + \sqrt{5}}}$

d. $5x^4 + x^3 - 2x^2 = 0$

$x^2(5x^2 + x - 2) = 0$

Both x^2 and $5x^2 + x - 2$ are quadratic equations, so you can solve them using any method.

$x^2 = 0$; $5x^2 + x - 2 = 0$

$x = \pm\sqrt{0}$; $a = 5$, $b = 1$, $c = -2$

$x = 0$; $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = 0$; $x = \frac{-1 \pm \sqrt{1^2 - 4(5)(-2)}}{2(5)}$

$x = 0$; $x = \frac{-1 \pm \sqrt{1^2 - 4(5)(-2)}}{10}$

$x = 0$; $x = \frac{-1 \pm \sqrt{1 + 41}}{10}$

$x = 0$; $x = \frac{-1 \pm \sqrt{41}}{10}$

$\boldsymbol{x = 0, \frac{-1 + \sqrt{41}}{10}, \frac{-1 - \sqrt{41}}{10}}$

Answers to Exercises:

1a. $x = -2, 2$; *2a.* $x = -3\sqrt{2}, 3\sqrt{2}$; *2b.* $x = -1, 0, \frac{2}{15}$;

2c. NO SOLUTIONS ; *2d.* $x = -\sqrt{2}, -\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}, \sqrt{2}$;

3a. $x = \sqrt{1 + \sqrt{11}}$, $-\sqrt{1 + \sqrt{11}}$; *4a.* NO SOLUTIONS ;

4b. $x = \sqrt{6 + \sqrt{33}}, -\sqrt{6 + \sqrt{33}}, \sqrt{6 - \sqrt{33}}, -\sqrt{6 - \sqrt{33}}$;

4c. $x = \sqrt{1 + \sqrt{5}}, -\sqrt{1 + \sqrt{5}}$; *4d.* $x = 0, -\frac{1}{10} + \frac{\sqrt{41}}{10}, -\frac{1}{10} - \frac{\sqrt{41}}{10}$;