Function:

- a connection between sets in which each element of the first set corresponds with **exactly** one element of the second set
  - the reverse is not necessarily true, but if it is, the function is one-to-one (more about these functions in Lesson 27)
- the first set is called the domain and the second set is called the range
  - an example of a function is the set of students in this classroom and their birthdays
    - each student has exactly one day when they were born, but that day could correspond with more than one student (function)
      - the set of students are the domain and the set of birthdays are the range
  - an another example of a function is the set of students at Purdue and their student ids
    - each student has exactly one student id and each student id corresponds with exactly one student (one-to-one function)
      - the set of students are the domain and the set of student ids are the range

Functions as equations:

- in terms of an equation, a function can be thought of as a rule that states what operations are performed on the inputs (the values that are plugged into a function)
  - for the function $f(x) = \frac{1-x^2}{3}$, we need to square each input, then negate the value, then add 1, and finally divide by 3
- the domain is the set of inputs and the range is the set of outputs
  - if I replace x with 0, then 0 is my input and it is an element of the domain; plugging 0 in for x results in $f(0) = \frac{1-0^2}{3}$, which simplifies to $f(0) = \frac{1}{3}$, so $\frac{1}{3}$ is an element of the range
- in order for an equation to be a function, every input can have only one output
- functions are represented by arbitrary letters, such as f, g, h ...

Domain:

- the set of inputs for which a function is defined, which means a meaningful output is produced
  - if an input makes a function undefined, that input is not part of the domain
  - for the function $f(x) = \frac{1-x^2}{3}$ from the previous page, $f(0) = \frac{1}{3}$, so that is an example of a function value that is defined

For many types of functions, the domain is unrestricted. This means that any real number can be plugged into the function, and a meaningful output will be produced. Linear functions (which we'll cover in Lesson 20) are one example, and quadratic functions (which we'll cover in Lessons 23 & 24) are another.

Linear Function	**Quadratic Function**
$f(x) = ax + b$ Domain: $(-\infty, \infty)$	$f(x) = ax^2 + bx + c$ Domain: $(-\infty, \infty)$
For each of these functions, the domain is all real numbers $(-\infty, \infty)$. This means we could plug in any real number from negative infinity to positive infinity, and we will get a real number back as our output. The reason this is the case is because there is nothing to restrict the domain of either of these functions. There are no fractions, so we don't have to worry about restricting a value that will result in a denominator of zero. And there are no square roots (or any other even roots), so we don't have to worry about restricting values that will result in a negative radicand. These are two types of restrictions we'll focus on going forward; later we'll also see how logarithms can restrict the domain of a function.	

As mentioned on the previous page, there are many types of functions that have unrestricted domains. However there are two types of functions (for now) with domains that are restricted. The first type of function we'll see that has a restricted domain is a function defined by a square root.

- **<u>square root functions</u>**
  - **<u>cannot take the square root of a negative value</u>**
  - if $g(x) = \sqrt{x}$, then $x \geq 0$ and the domain is all real numbers greater than or equal to 0

Domain of g in interval notation: : $[0, \infty)$

For a function such as $g(x) = \sqrt{x}$, x cannot be negative because it is impossible to take the even root of a negative number and get a real number in return ($x \geq 0$). That means x can equal 0, or be larger than 0; this is expressed using interval notation as $[0, \infty)$. The square bracket on 0 indicates that 0 is part of the interval, so 0 and any number bigger than 0 can be used as an input for the function $g(x) = \sqrt{x}$ and they will all produce meaningful outputs (they will not make the function undefined).

<u>Example 1:</u> Find the domain of each of the following functions, and list in interval notation. **<u>Use a number line to convert from inequalities to intervals, if necessary.</u>**

a. $f(x) = \sqrt{2x + 7}$

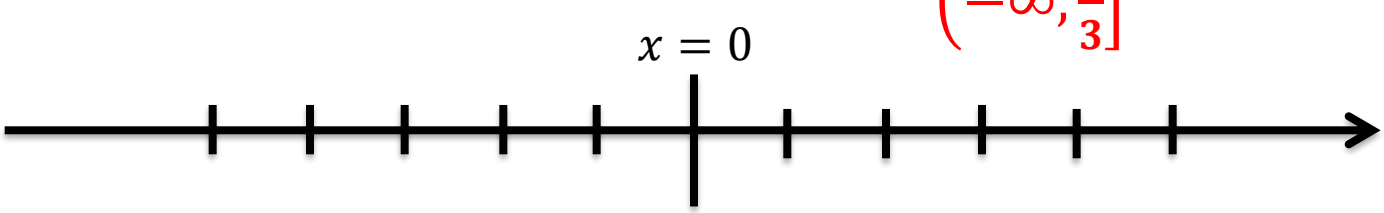
b. $g(x) = \sqrt{8 - 3x}$

$8 - 3x \geq 0$

$-3x \geq -8$

$x \leq \frac{8}{3}$

$\left(-\infty, \mathbf{\frac{8}{3}}\right]$



c. $h(x) = 2 + \sqrt{3 + x}$

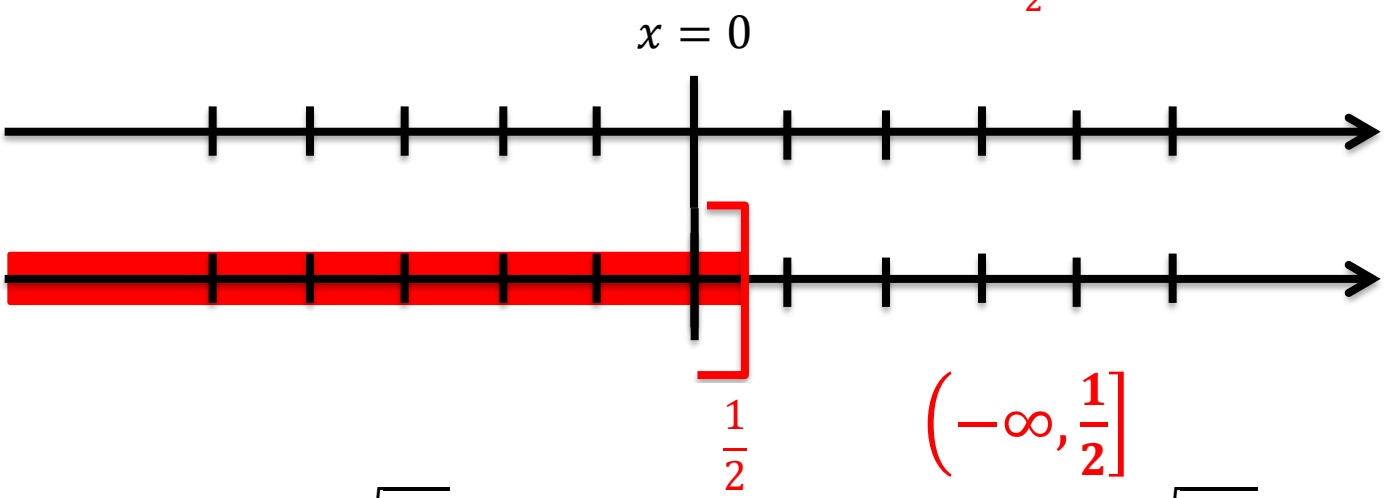
d. $j(x) = 1 - \sqrt{1 - 2x}$

$1 - 2x \geq 0$

$1 \geq 2x$

$\frac{1}{2} \geq x$

$x \leq \frac{1}{2}$



$\left(-\infty, \frac{1}{2}\right]$

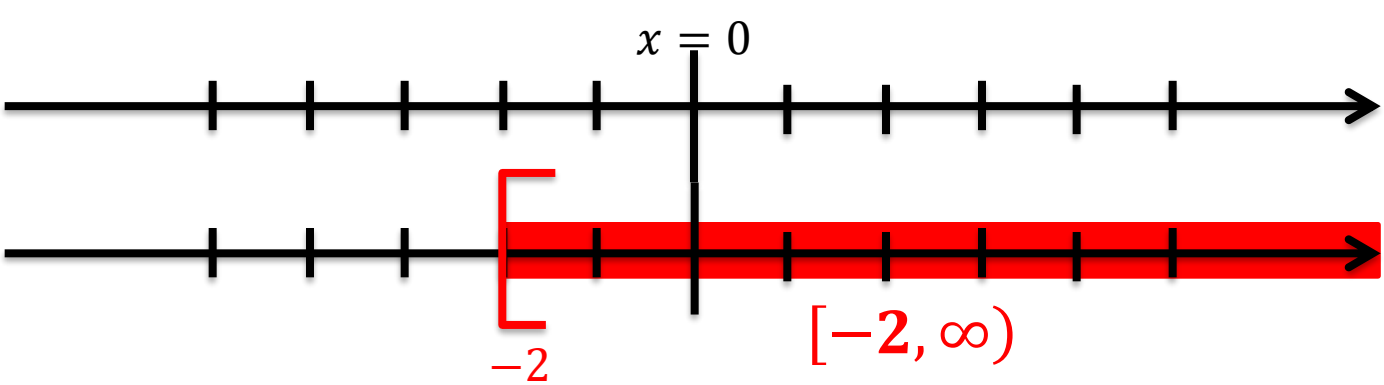
e. $k(x) = 5 + \sqrt{\frac{4-x}{3}}$

f. $l(x) = 3 - \sqrt{\frac{x+2}{5}}$

$\frac{x+2}{5} \geq 0$

$x + 2 \geq 0$

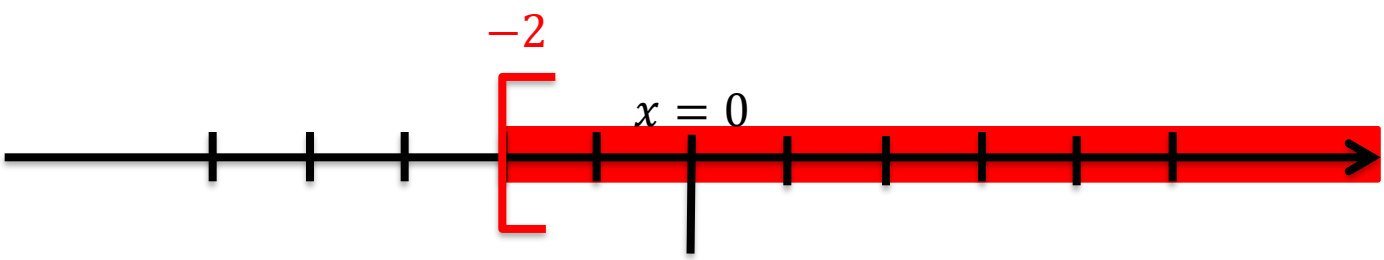
$x \geq -2$



$[-2, \infty)$

Keep in mind that anytime you see a function containing a square root, the radicand of that square root must be non-negative (equal to zero or greater than zero).

Also, keep in mind that intervals must go in order from smallest to largest when going from left to right, just like a number line. On Example 1 part f., I simplified the inequality $\frac{x+2}{5} \geq 0$ to get $x \geq -2$. In order to convert this inequality to interval, I used a number line:

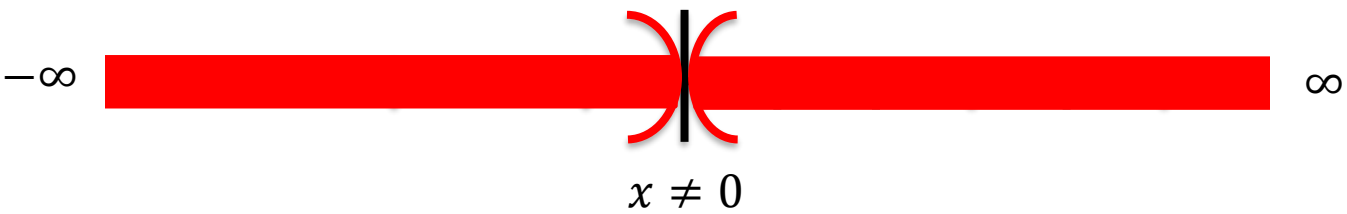


This is how I came up with my answer in interval notation, $[\mathbf{-2}, \infty)$. Since -2 is to the left of infinity on the number line, -2 must be to the left of infinity in my interval. Be aware that expressing the same answer in the opposite order, $(\infty, -2]$, will result in an incorrect answer. You not only need the correct numbers and the correct symbols, you need them in the correct order.

Remember that just like a number line, an interval should always go from smallest to largest when going from left to right.

We have seen how functions defined by square roots have restrictions on their domains, and now we'll look at rational functions. A rational function is simply a function defined by a fraction. Remember that the denominator of a fraction cannot equal zero.

- **<u>rational functions (fractions)</u>**
  - **<u>denominator cannot equal zero</u>**
  - if $f(x) = \frac{1}{x}$, then $x \neq 0$ and the domain is all real numbers except 0



Domain of f in interval notation: $(-\infty, 0) \cup (0, \infty)$

For a function such as $f(x) = \frac{1}{x}$, x cannot be 0 because we can never divide by 0 ($x \neq 0$). That means x can less than 0 ($x < 0$) or x can be greater than 0 ($x > 0$), but it cannot equal 0. This is expressed using interval notation as $(-\infty, 0) \cup (0, \infty)$; the interval $(-\infty, 0)$ represents the set of negative numbers ($x < 0$) and the interval $(0, \infty)$ represents the set of positive numbers ($x > 0$). Notice that in both intervals, $(-\infty, 0)$ and $(0, \infty)$, there is a parenthesis on 0. That means that 0 is not part of either interval. For the function $f(x) = \frac{1}{x}$ we can choose inputs that are as close to zero as we'd like, but cannot use 0 as an input because it will produce a real number as an output (meaning it makes the function undefined).

<u>Example 2:</u> Find the domain of each of the following functions, and list in interval notation. **<u>Use a number line to convert from inequalities to intervals, if necessary.</u>**

a. $m(x) = \frac{x}{x-4}$

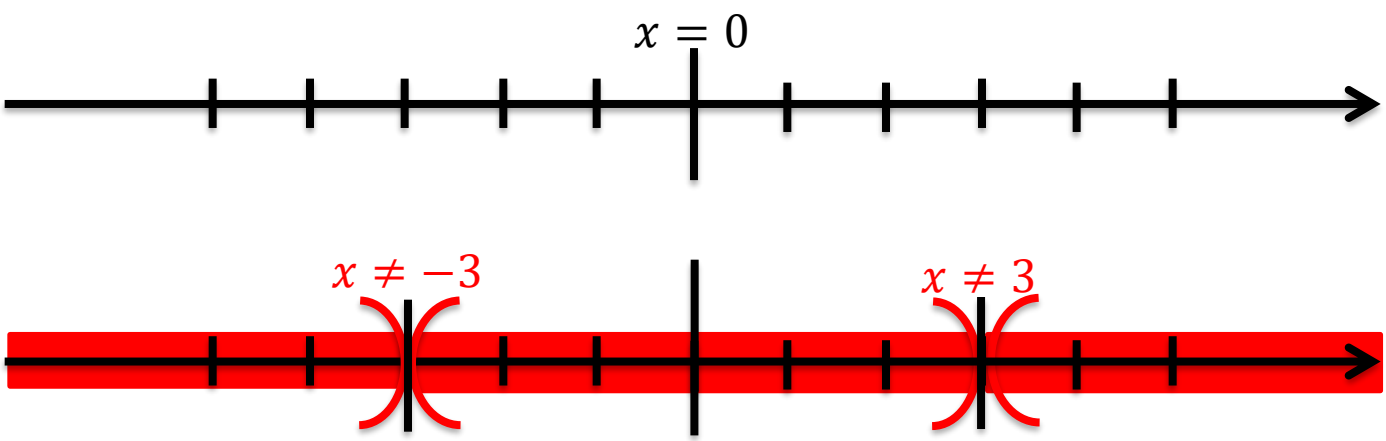
b. $n(x) = \frac{x+1}{x^2-9}$

$x^2 - 9 \neq 0$

$x^2 \neq 9$

$x \neq \pm 3$

$x \neq -3$ and $x \neq 3$



$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Remember that for now, there are only two types of functions that could have a restricted domain; fractions and square roots. When you see a fraction, state that the denominator cannot equal zero, and when you see a square root, state that the radicand (the expression under the square root symbol) cannot be negative:

Denominators $\neq 0$ **Radicands ≥ 0**

<u>Example 3:</u> Find the domain of each of the following functions, and list in interval notation. **<u>Use a number line to convert from inequalities to intervals, if necessary.</u>**

a. $p(x) = \frac{\sqrt{x}}{x-1}$

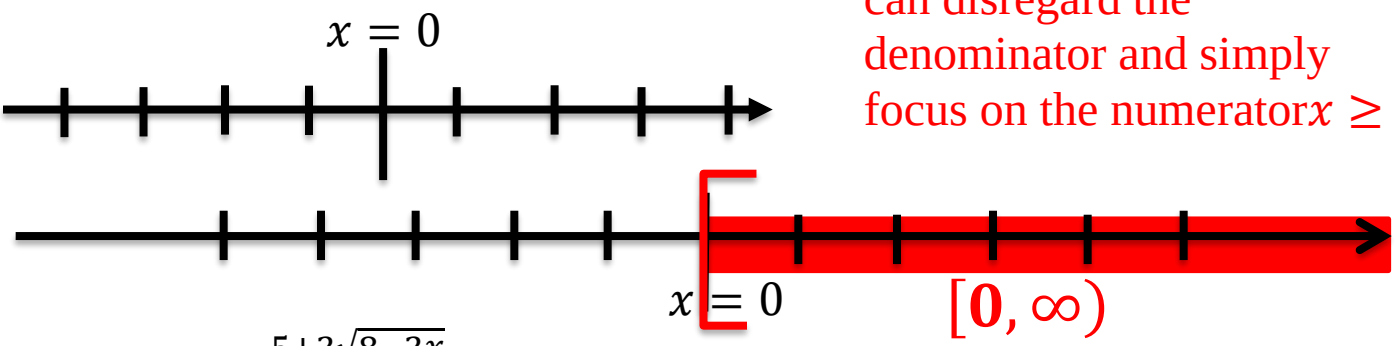
b. $q(x) = \frac{1-\sqrt{2x}}{x^2+1}$

$2x \geq 0 \quad \text{AND} \quad x^2 + 1 \neq 0$

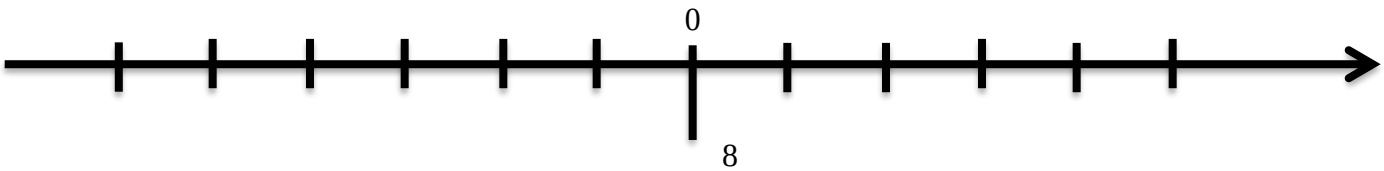
$x \geq 0 \quad \text{AND} \quad x^2 \neq -1$

$x \geq 0 \quad \text{AND} \quad x \neq \pm\sqrt{-1}$

Since $x \neq \pm\sqrt{-1}$ does not exist with real numbers, we can disregard the denominator and simply focus on the numerator$x \geq 0$.



c. $r(x) = \frac{5+2\sqrt{8-2x}}{x^2+5x+6}$



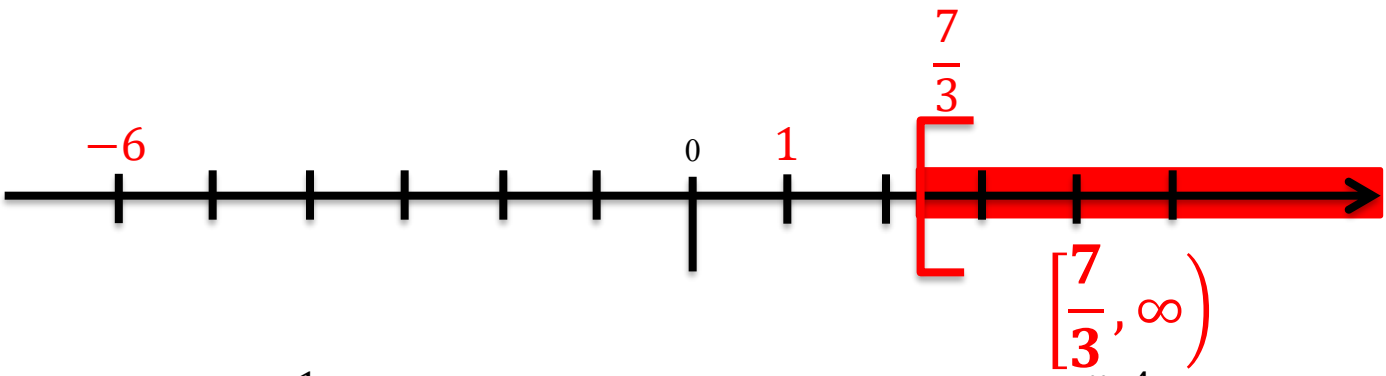
d. $s(x) = \frac{\sqrt{3x-7}}{x^2+5x-6}$

$$3x - 7 \geq 0 \text{ AND } x^2 + 5x - 6 \neq 0$$

$$3x \geq 7 \text{ AND } (x+6)(x-1) \neq 0$$

$$x \geq \frac{7}{3} \text{ AND } x + 6 \neq 0 \text{ AND } x - 1 \neq 0$$

$$x \geq \frac{7}{3} \text{ AND } x \neq -6 \text{ AND } x \neq 1$$



e. $f(x) = \frac{1}{\sqrt{x+3}}$

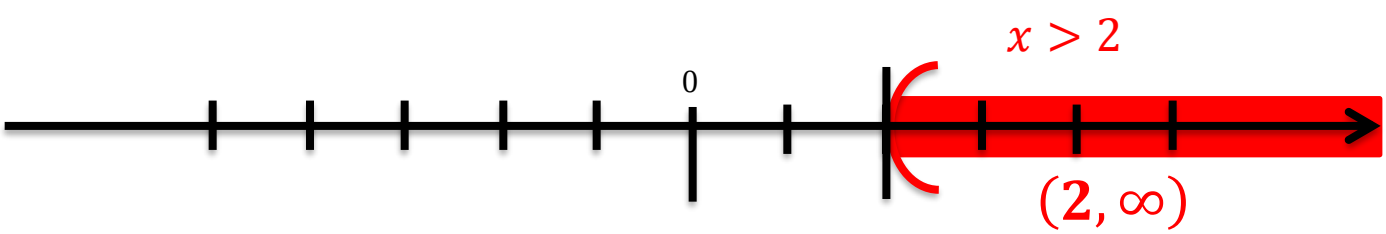
f. $g(x) = \frac{x-4}{\sqrt{x-2}}$

$$\sqrt{x-2} \neq 0 \text{ AND } x - 2 \geq 0$$

$$x - 2 \neq 0 \text{ AND } x \geq 2$$

$$x \neq 2 \text{ AND } x \geq 2$$

At this point we have two inequalities that are conflicting. The inequality on the left states that x cannot equal 2 ($x \neq 2$) and the inequality on the right states that x must be greater than or equal to 2 ($x \geq 2$). Since replacing x with 2 would result in the denominator of the original function being 0, 2 is not a valid input. So I will replace my two inequalities with just one, which states that x must be strictly greater than 2 ($x > 2$).



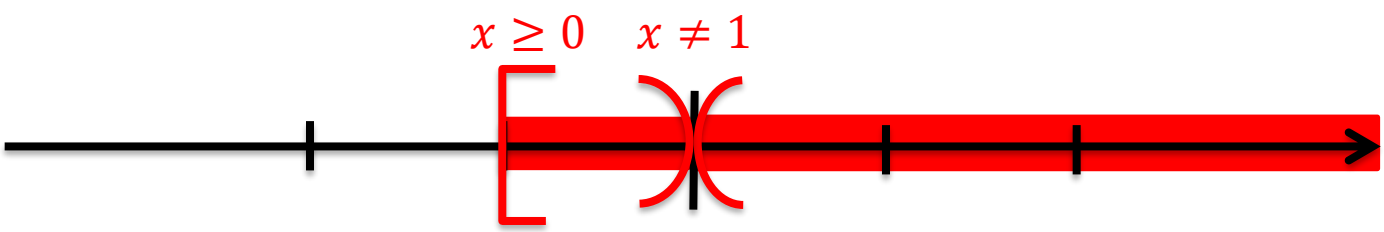
g. $h(x) = \frac{x}{\sqrt{x}-1}$

h. $j(x) = \frac{\sqrt{4-x}}{2-\sqrt{x}}$

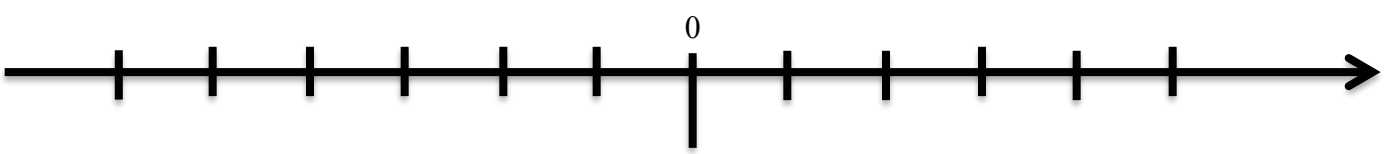
$x \geq 0$ AND $\sqrt{x} - 1 \neq 0$

$x \geq 0$ AND $\sqrt{x} \neq 1$

$x \geq 0$ AND $x \neq 1$



$[\mathbf{0}, \mathbf{1}) \cup (\mathbf{1}, \infty)$



Answers to Examples:

1a. $\left[-\frac{7}{2}, \infty\right)$; *1b.* $\left(-\infty, \frac{8}{3}\right]$; *1c.* $[-3, \infty)$; *1d.* $\left(-\infty, \frac{1}{2}\right]$;
1e. $(-\infty, 4]$; *1f.* $[-2, \infty)$; *2a.* $(-\infty, 4) \cup (4, \infty)$;
2b. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$; *3a.* $[0, 1) \cup (1, \infty)$; *3b.* $[0, \infty)$;
3c. $(-\infty, -3) \cup (-3, -2) \cup (-2, 4]$; *3d.* $\left[\frac{7}{3}, \infty\right)$; *3e.* $(-3, \infty)$;
3f. $(2, \infty)$; *3g.* $[0, 1) \cup (1, \infty)$; *3h.* $[0, 4)$;