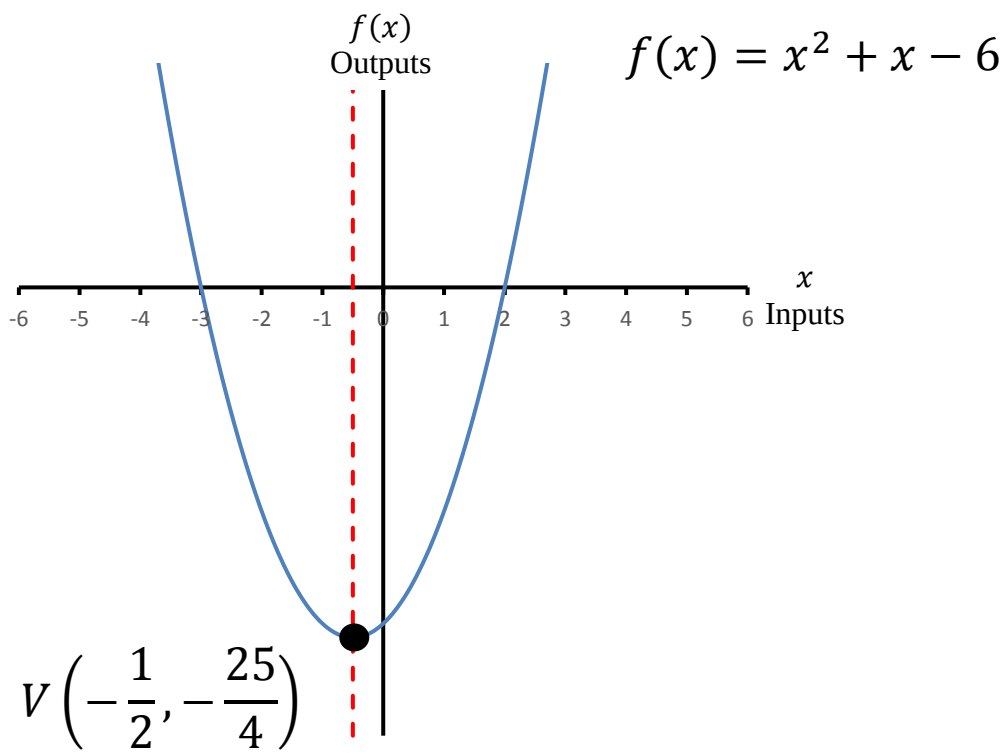
Quadratic Functions:

- functions defined by quadratic expressions $(ax^2 + bx + c)$
  - the degree of a quadratic function is **ALWAYS** 2
- the most common way to write a quadratic function (and the way we have seen quadratics in the past) is polynomial form
  - $f(x) = ax^2 + bx + c$
- the graph of a quadratic function is a parabola $(\cup$ or $\cap)$
  - in order to be the graph of a function, the parabola must be vertical
    - a parabola opening sideways would not pass the vertical line test and as a result would not be a function
  - the leading coefficient a determines whether the parabola opens up or down
    - if $a > 0$ the parabola opens up $(\cup)$
    - if $a < 0$ the parabola opens down $(\cap)$

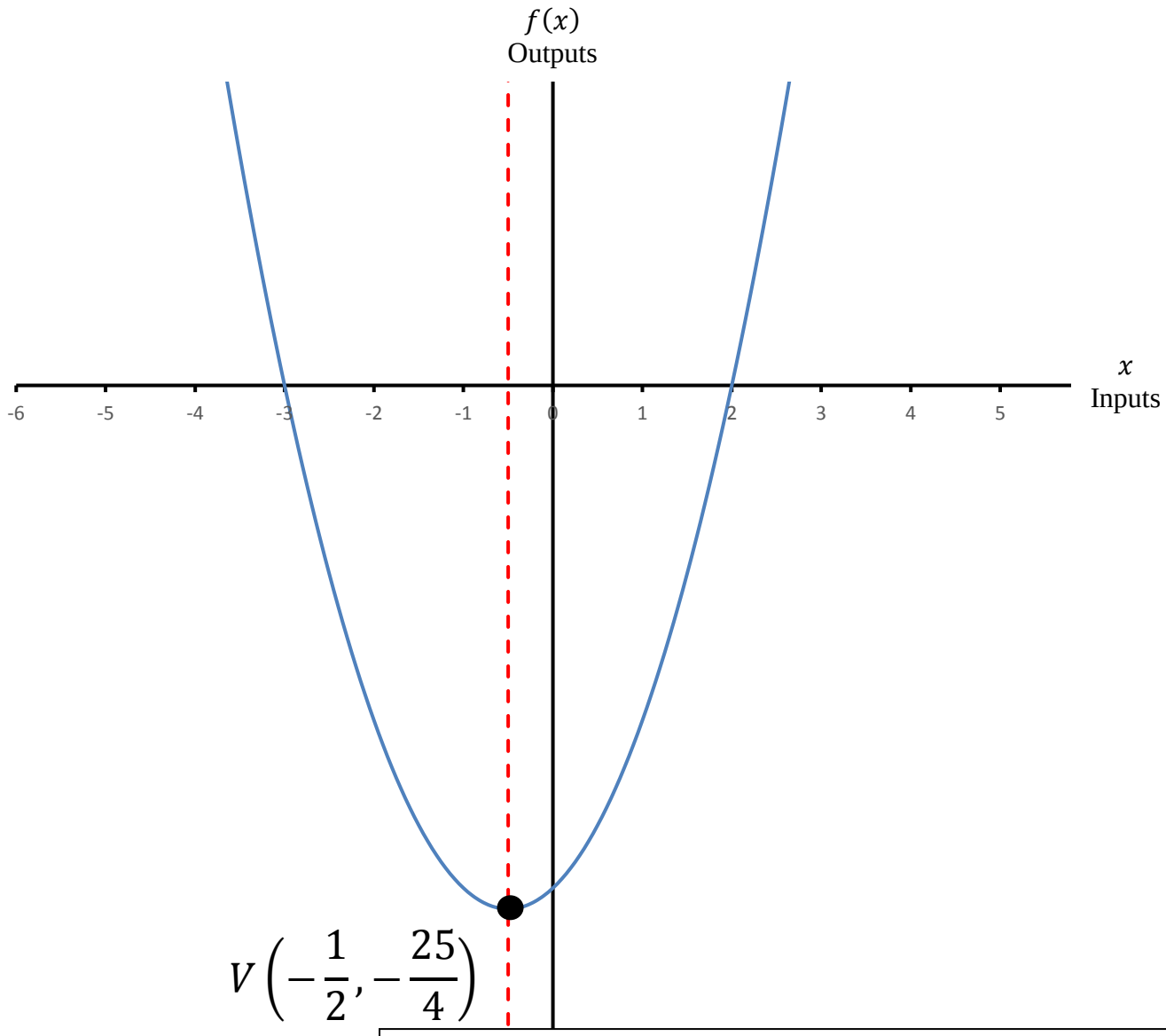
Vertex:

- the turning point of a parabola

Example 1: Given below is the graph of the quadratic function f. Use the function and its graph to find the following:



a. Find the domain (the set of inputs or x-values) and range (the set of outputs or function values) of the function $f(x) = x^2 + x - 6$.



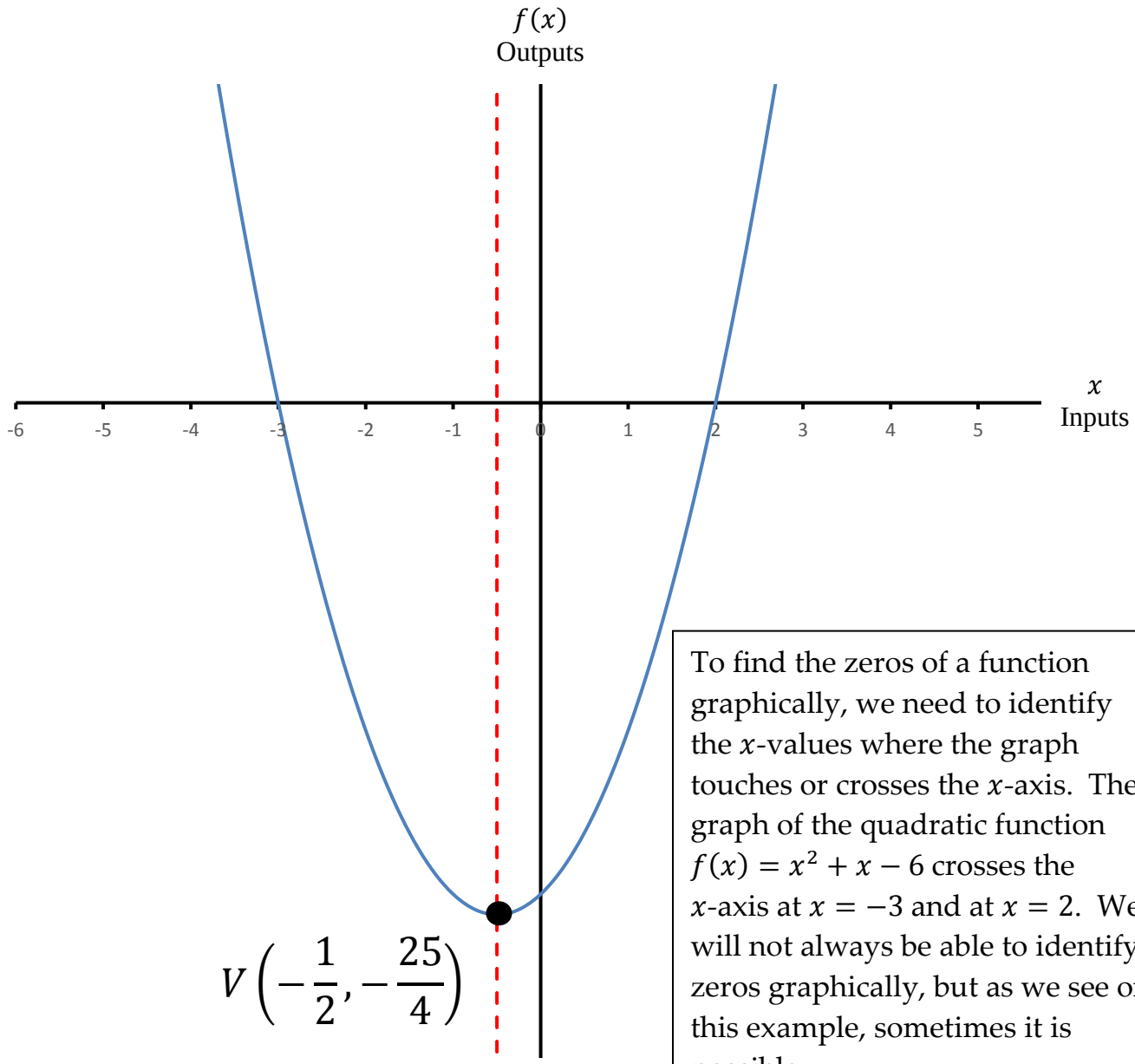
Domain: $(-\infty, \infty)$

Range: $\left[-\frac{25}{4}, \infty\right)$

The domain can found algebraically or graphically. Algebraically we can see that the quadratic function $f(x) = x^2 + x - 6$ has no square roots and/or fractions, so there is nothing restricting its domain. Graphically we can see the that the parabola continues to move to the left $(-\infty)$ and to the right (∞), so the domain goes on forever in both directions.

The range can be found graphically as well; we can see that the smallest output of the function is the y-coordinate of the vertex $\left(-\frac{25}{4}\right)$, and from the graph just continues to rise to infinity.

b. Find the zeros of the function $f(x) = x^2 + x - 6$.
($f(x) = 0$ when $x = ?$)

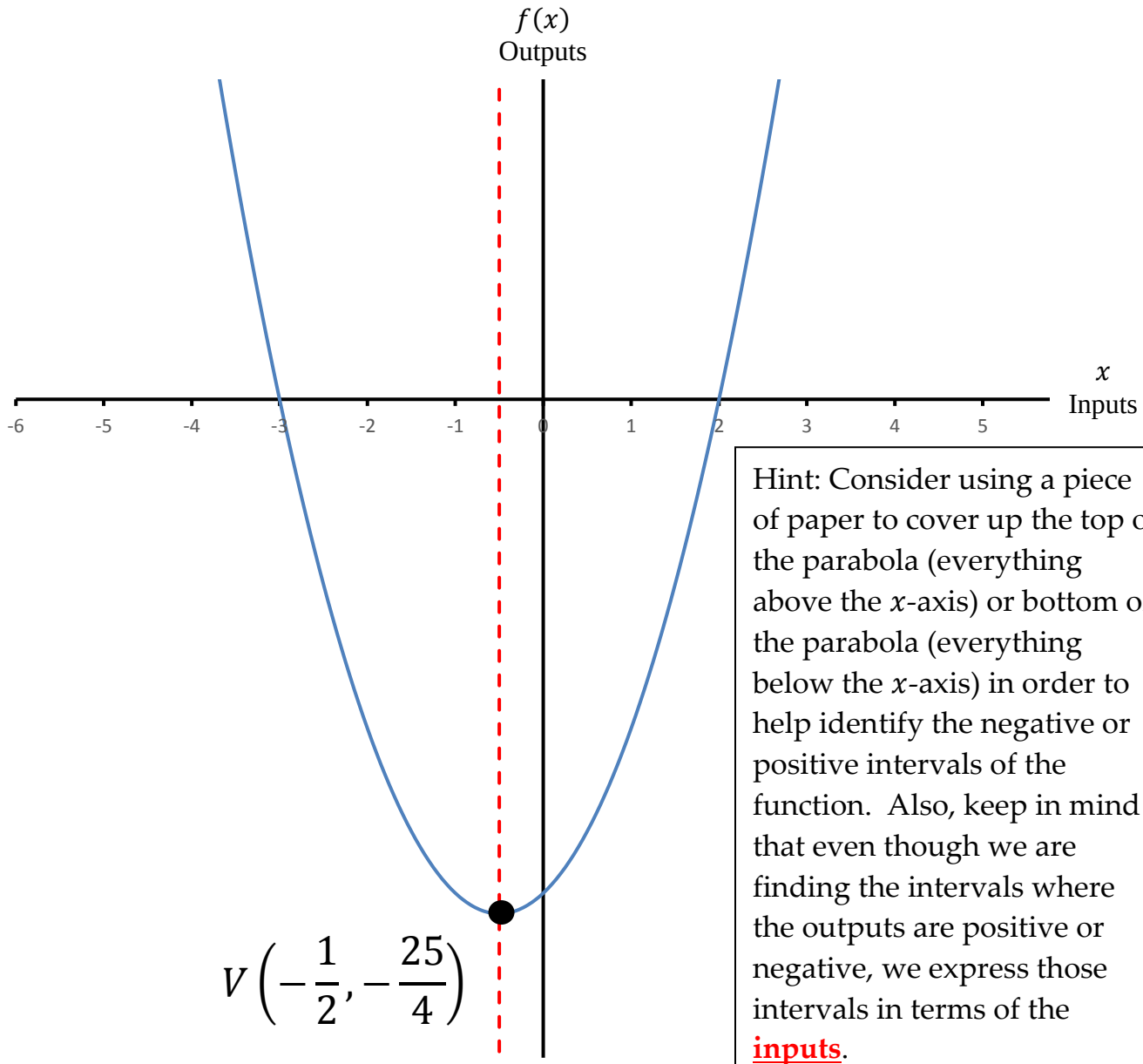


$f(x) = 0$ when $x = -3, 2$

To find the zeros of a function graphically, we need to identify the x-values where the graph touches or crosses the x-axis. The graph of the quadratic function $f(x) = x^2 + x - 6$ crosses the x-axis at $x = -3$ and at $x = 2$. We will not always be able to identify zeros graphically, but as we see on this example, sometimes it is possible.

Another option is to find the zeros of a function algebraically, which simply means replacing $f(x)$ with 0 and solving the quadratic equation. In this case, the quadratic equation would be $0 = x^2 + x - 6$, and we'd still come up with $x = -3, 2$. To review how to solve quadratic equations, take a look at Lessons 12, 13, and 14.

c. List the positive and negative intervals of $f(x) = x^2 + x - 6$.
What $x -$ values make $f(x) > 0$? What $x -$ values make $f(x) < 0$?



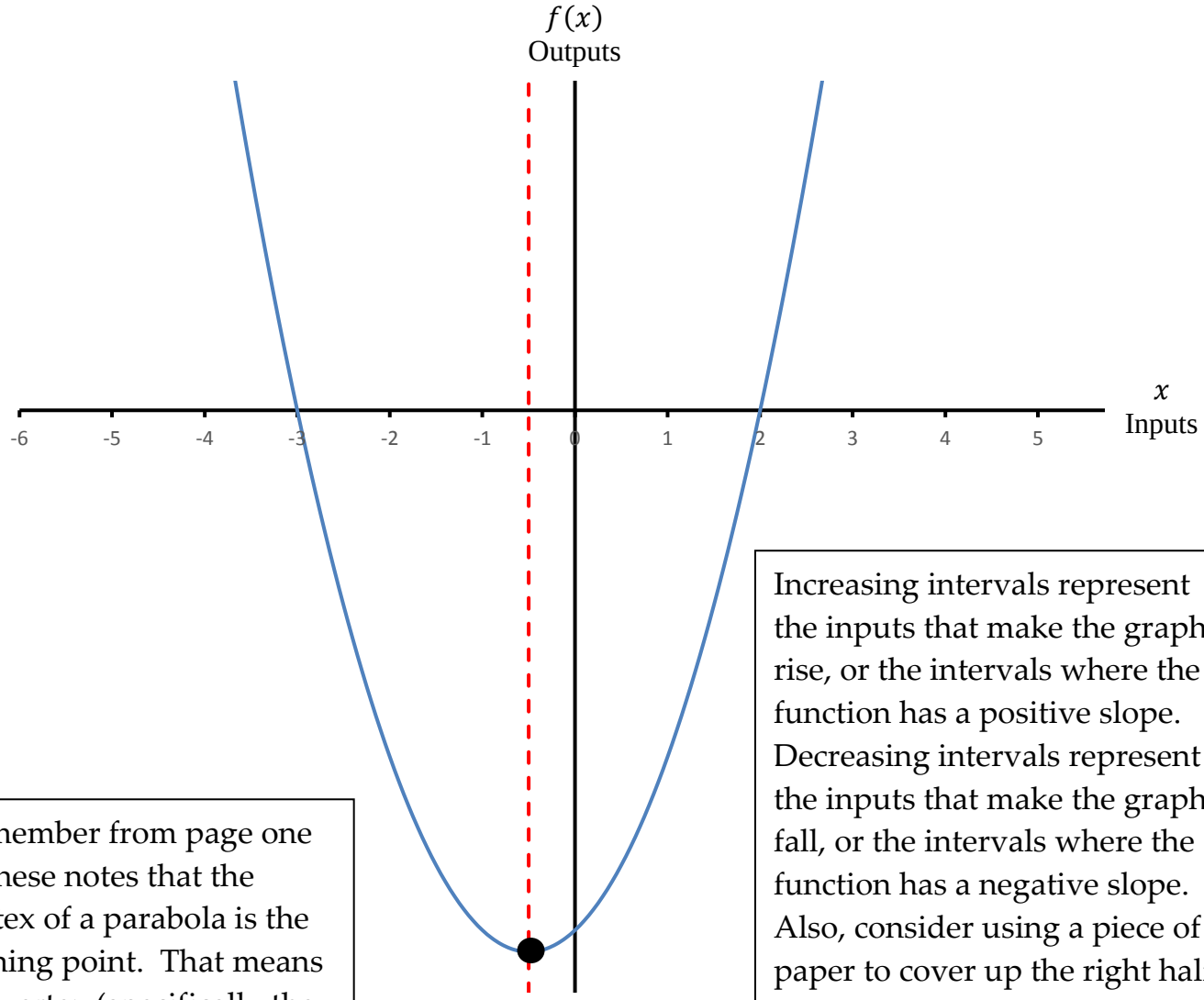
Hint: Consider using a piece of paper to cover up the top of the parabola (everything above the x-axis) or bottom of the parabola (everything below the x-axis) in order to help identify the negative or positive intervals of the function. Also, keep in mind that even though we are finding the intervals where the outputs are positive or negative, we express those intervals in terms of the **inputs**.

The inputs ($x -$ values) that make $f(x) > 0$ are $(-\infty, -3) \cup (2, \infty)$

The inputs ($x -$ values) that make $f(x) < 0$ are $(-3, 2)$

Be aware that an answer such as $(-3, 2)$ could be an interval, as it was here, or it could also be an ordered, such a quadratic function with a vertex at the point $(-3, 2)$. This ambiguity sometimes causes confusion when working with quadratic functions.

d. List the increasing and decreasing intervals of $f(x) = x^2 + x - 6$.
What $x -$ values make $f(x)$ rise? What $x -$ values make $f(x)$ fall?



$V\left(-\frac{1}{2}, -\frac{25}{4}\right)$

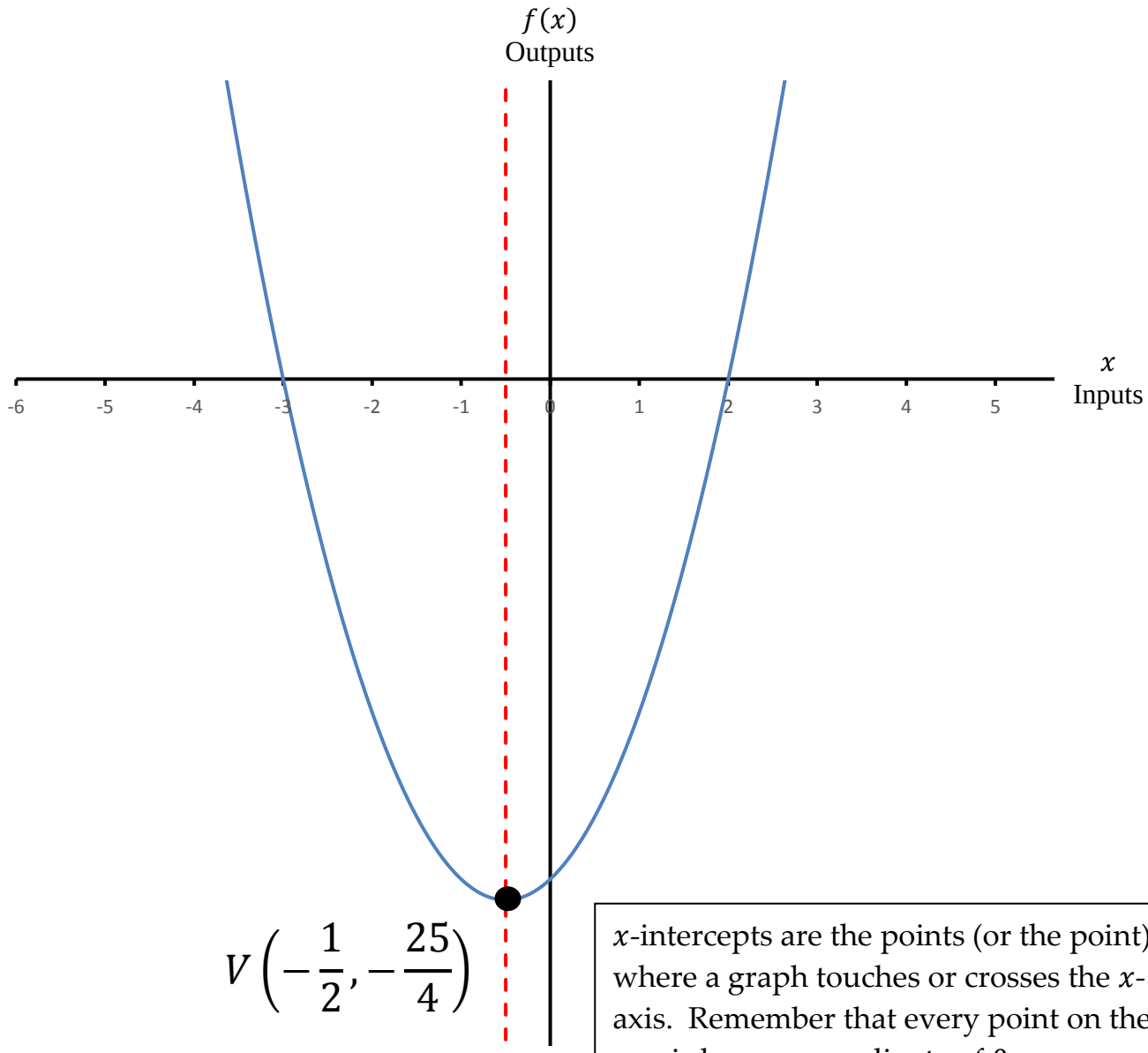
Remember from page one of these notes that the vertex of a parabola is the turning point. That means the vertex (specifically the x-coordinate of the vertex) will always be where a parabola turns from decreasing to increasing (as we see on this example) or increasing to decreasing.

Increasing intervals represent the inputs that make the graph rise, or the intervals where the function has a positive slope. Decreasing intervals represent the inputs that make the graph fall, or the intervals where the function has a negative slope. Also, consider using a piece of paper to cover up the right half (everything to the left of the vertex) or left half (everything to the right of the vertex) of the parabola in order to help identify the decreasing or increasing intervals of the function.

The inputs ($x -$ values) that make $f(x)$ rise are $\left(-\frac{1}{2}, \infty\right)$

The inputs ($x -$ values) that make $f(x)$ fall are $\left(-\infty, -\frac{1}{2}\right)$

e. List the intercepts of the function $f(x) = x^2 + x - 6$ as order pairs (remember that intercepts are points).



$x-$ intercepts: $(-3, 0), (2, 0)$

$y-$ intercepts: $(0, -6)$

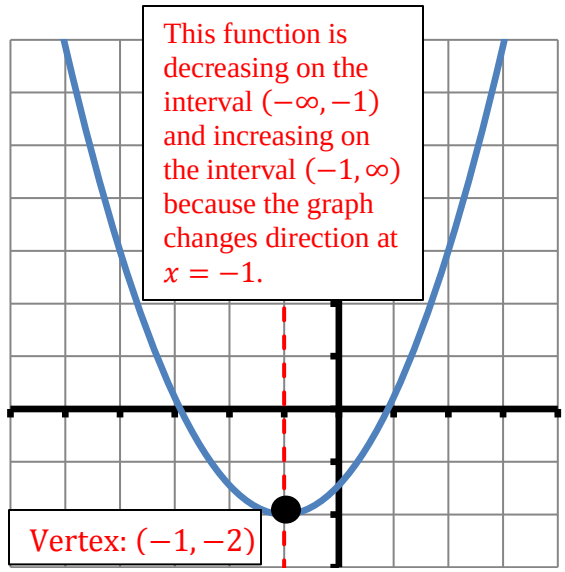
x-intercepts are the points (or the point) where a graph touches or crosses the x-axis. Remember that every point on the x-axis has a y-coordinate of 0.

y-intercepts are the points (or the point) where a graph touches or crosses the y-axis. Remember that every point on the y-axis has an x-coordinate of 0.

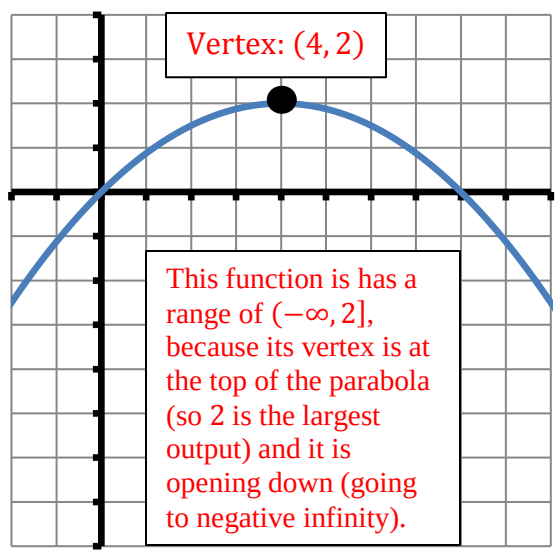
Be sure to always express intercepts as ordered pairs.

What the Vertex Tells Us About a Quadratic Function:

- the x-coordinate of the vertex determines the where the graph changes from increasing to decreasing or decreasing to increasing
  - if the parabola opens up, the graph is decreasing from $(-\infty, x)$ and increasing from (x, ∞)
  - if the parabola opens down, the graph is increasing from $(-\infty, x)$ and decreasing from (x, ∞)



- the x-coordinate of the vertex also determines the line of symmetry
  - the line of symmetry is the vertical line that cuts the parabola in half
  - the equation of the line of symmetry is $x = x$-coordinate
- the y-coordinate of the vertex is either the maximum function value or minimum function value (largest or smallest **output**)
  - if a parabola opens up, the vertex is a minimum and the y-coordinate is smallest function value
  - if a parabola opens down, the vertex is a maximum and the y-coordinate is largest function value

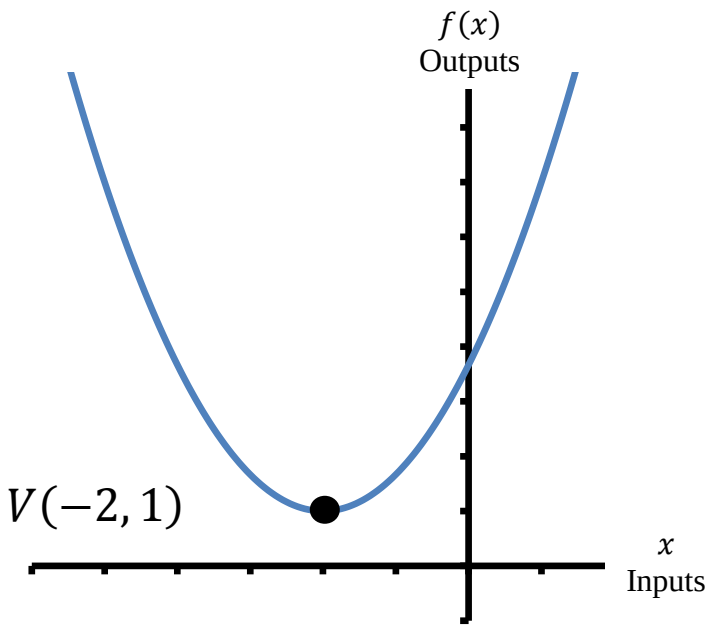


- the y-coordinate of the vertex also determines the range of a quadratic function
  - if the vertex is the minimum function value, the range is the y-coordinate to infinity $[y, \infty)$ because the parabola opens up
  - if the vertex is the maximum function value, the range is negative infinity to the y-coordinate $(-\infty, y]$ because the parabola opens down

If the graph of a quadratic function does not touch or cross the x-axis, then the function has no zeros and no x-intercepts. Examples of this are given below. Even without the graphs provided, we could still determine that the functions are never equal to zero by setting each expression equal to zero and solving. In each case we will end up with imaginary solutions, which would indicate that there are no real zeros.

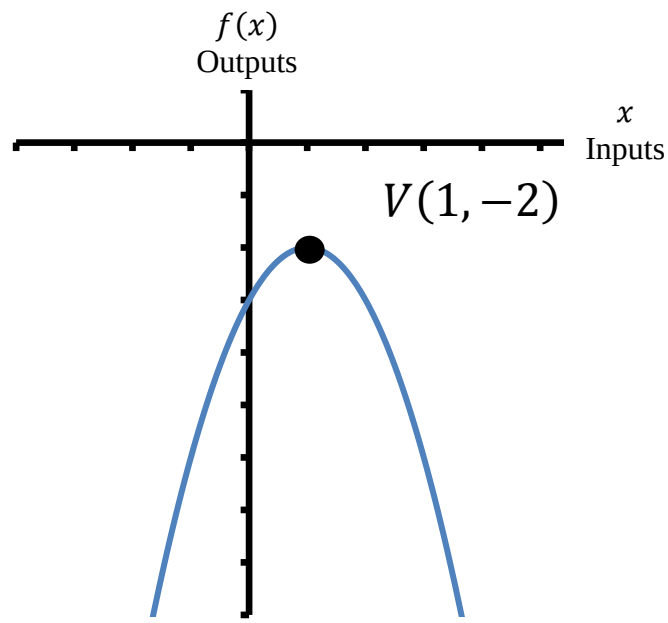
Notice that these quadratic functions are either always positive or always negative. If the graph of the function does not cross the x-axis, the sign of the function (positive or negative) does not change.

$$f(x) = \frac{2}{3}x^2 + \frac{8}{3}x + \frac{11}{3}$$



Notice that the graph of the function f never touches or crosses the x-axis. Therefore there are no zeros, no x-intercepts, and no negative intervals, because $f(x) > 0$ for every input.

$$g(x) = -x^2 + 2x - 3$$



The graph of the function g never touches or crosses the x-axis because $g(x) < 0$ for every input.

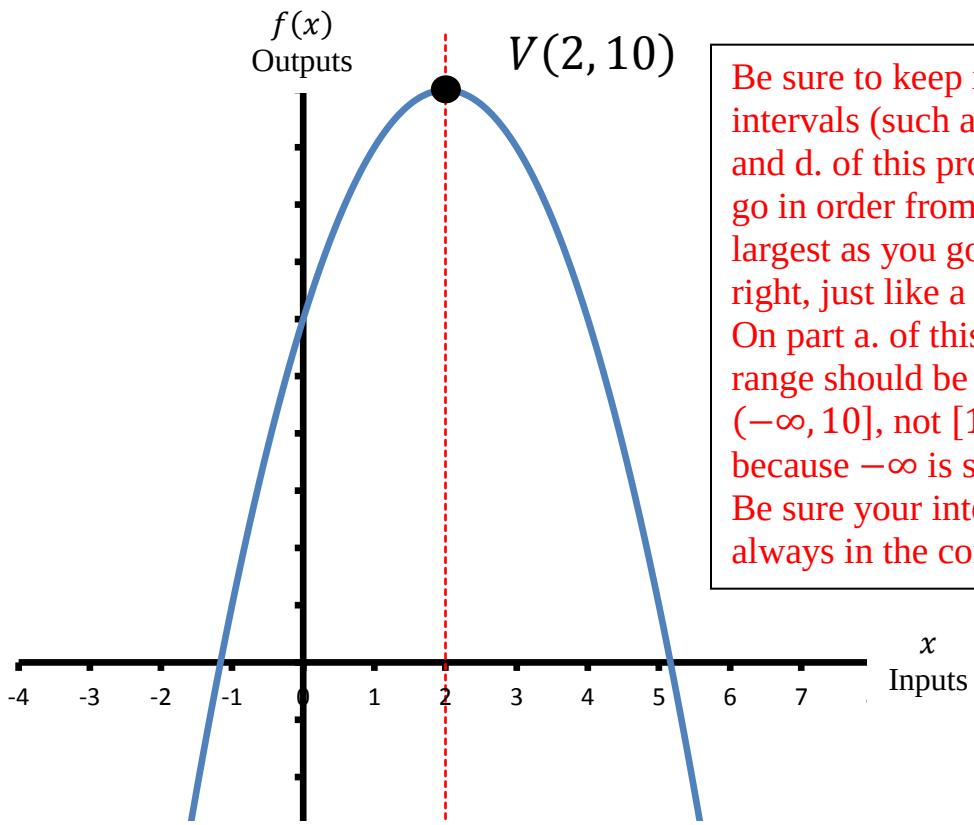
Zeros: NONE

$x -$ intercepts: NONE

$g(x) > 0$: NONE

Example 2: Given below is the graph of the quadratic function g. Use the function and its graph to find the following:

$$g(x) = -x^2 + 4x + 6$$



Be sure to keep in mind that intervals (such as parts a., c., and d. of this problem) always go in order from smallest to largest as you go from left to right, just like a number line. On part a. of this Example, the range should be listed as $(-\infty, 10]$, not $[10, -\infty)$, because $-\infty$ is smaller than 10. Be sure your intervals are always in the correct order.

a. Find the domain (the set of inputs or x-values) and range (the set of outputs or function values) of the function $g(x) = -x^2 + 4x + 6$.

Domain : $(-\infty, \infty)$ **Range :** $(-\infty, \mathbf{10}]$

b. Find the zeros of the function ($g(x) = 0$ when $x = ?$).

$0 = -x^2 + 4x + 6$

$x^2 - 4x = 6$

$x^2 - 4x + 4 = 6 + 4$

$(x - 2)^2 = 10$

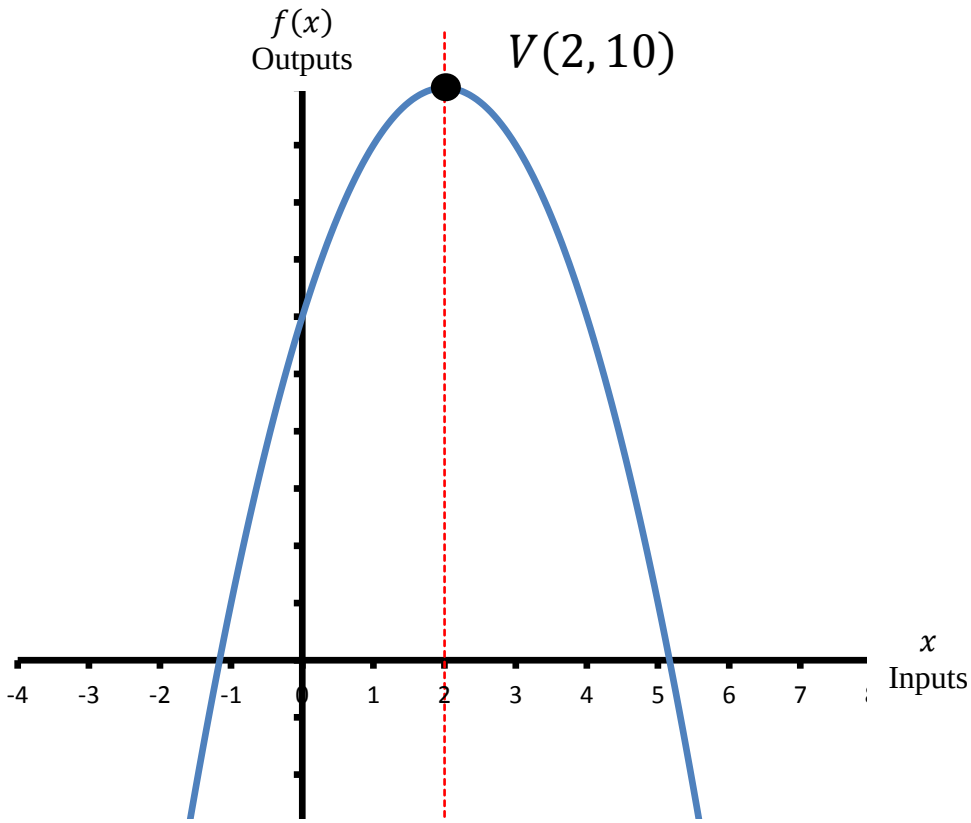
$\boldsymbol{x = 2 \pm \sqrt{10}}$

I solve the quadratic equation $0 = -x^2 + 4x + 6$ by completing the square, however you could also solve by using the quadratic formula. Regardless, the answers should be the same, $x = 2 - \sqrt{10}$, $2 + \sqrt{10}$.

If you need to review the steps for solving quadratic equations by completing the square or using the quadratic formula, take a look at Lessons 13 or 14.

This quadratic function has two zeros, $\boldsymbol{x = 2 - \sqrt{10}}$ (which is approximately -1.2) and $\boldsymbol{x = 2 + \sqrt{10}}$ (which is approximately 5.2). This is consistent with the graph given above, which shows that the parabola crosses the x-axis just to the left of -1 and just to the right of 5.

$$g(x) = -x^2 + 4x + 6$$



c. List the positive and negative intervals of $g(x) = -x^2 + 4x + 6$.
What $x -$ values make $g(x) > 0$? What $x -$ values make $g(x) < 0$?

$+ : \left(\mathbf{2 - \sqrt{10}, 2 + \sqrt{10}}\right)$ $\qquad$ $- : \left(\mathbf{-\infty, 2 - \sqrt{10}}\right) \cup \left(\mathbf{2 + \sqrt{10}, \infty}\right)$

d. List the increasing and decreasing intervals of $g(x) = -x^2 + 4x + 6$
What $x -$ values make $g(x)$ rise? What $x -$ values make $g(x)$ fall?

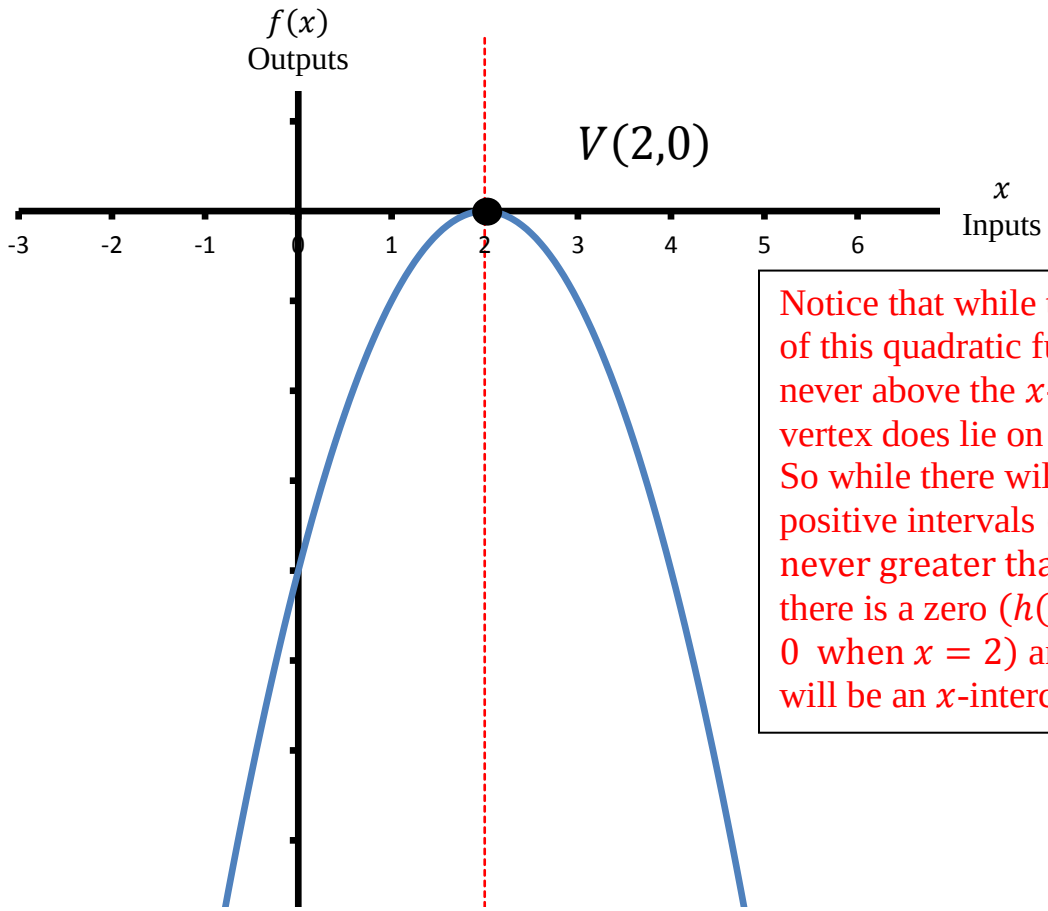
$\uparrow : (-\infty, \mathbf{2})$ $\qquad$ $\downarrow : (\mathbf{2}, \infty)$

e. List the intercepts of the function as order pairs (remember that intercepts are points).

$\boldsymbol{x -}$ **intercepts**: $\left(\mathbf{2 - \sqrt{10}, 0}\right), \left(\mathbf{2 + \sqrt{10}, 0}\right)$; $\boldsymbol{y -}$ **intercept**: $(\mathbf{0, 6})$

Example 3: Given below is the graph of the quadratic function h. Use the function and its graph to find the following:

$$h(x) = -x^2 + 4x - 4$$



Notice that while the graph of this quadratic function is never above the x-axis, the vertex does lie on the x-axis. So while there will be no positive intervals ($h(x)$ is never greater than zero), there is a zero ($h(x) = 0$ when $x = 2$) and there will be an x-intercept $(2, 0)$.

a. Find the domain (the set of inputs or x-values) and range (the set of outputs or function values) of the function $h(x) = -x^2 + 4x - 4$.

$D: (-\infty, \infty), R: (-\infty, 0]$

b. Find the zeros of the function ($h(x) = 0$ when $x = ?$).

$h(x) = 0$ when $x = 2$

c. List the positive and negative intervals of.
What $x -$ values make $h(x) > 0$? What $x -$ values make $h(x) < 0$?

$h(x) > 0$: NONE, $h(x) < 0: (-\infty, 2) \cup (2, \infty)$

d. List the increasing and decreasing intervals of.
What $x -$ values make $h(x)$ rise? What $x -$ values make $h(x)$ fall?

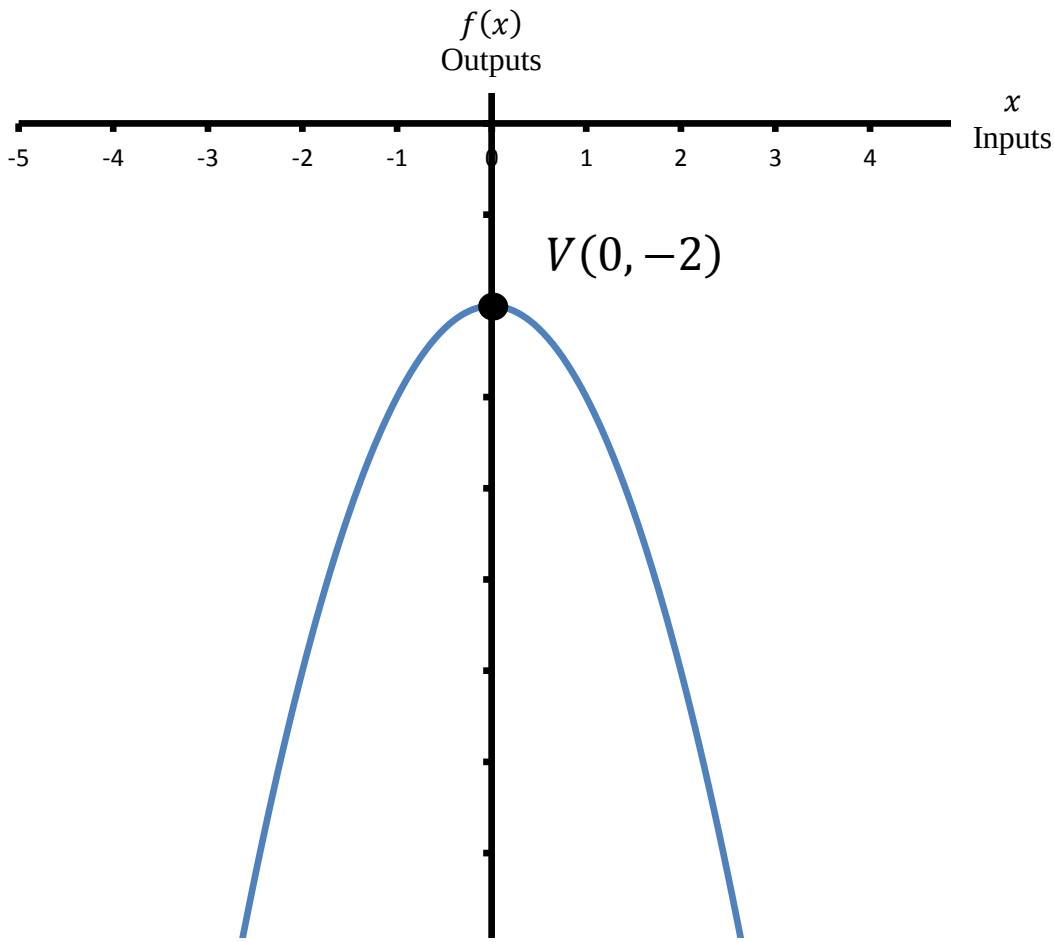
$\uparrow: (-\infty, 2), \downarrow: (2, \infty)$

e. List the intercepts of the function as order pairs (remember that intercepts are points).

$x -$ int: $(2, 0)$, $y -$ int: $(0, -4)$

<u>Example 4:</u> Given below is the graph of the quadratic function j. Use the function and its graph to find the following:

$$j(x) = -x^2 - 2$$



a. Find the domain (the set of inputs or x-values) and range (the set of outputs or function values) of the function $j(x) = -x^2 + 4x - 4$.

b. Find the zeros of the function $(j(x) = 0 \text{ when } x = ?)$.

c. List the positive and negative intervals of.
What $x -$ values make $j(x) > 0$? What $x -$ values make $j(x) < 0$?

d. List the increasing and decreasing intervals of.
What $x -$ values make $j(x)$ rise? What $x -$ values make $j(x)$ fall?

e. List the intercepts of the function as order pairs (remember that intercepts are points).

Answers to Examples:

1a. $D: (-\infty, \infty), R: \left[-\frac{25}{4}, \infty\right)$; *1b.* $f(x) = 0$ when $x = -3, 2$;

1c. $f(x) > 0: (-\infty, -3) \cup (2, \infty)$, $f(x) < 0: (-3, 2)$, ;

1g. $\uparrow: \left(-\frac{1}{2}, \infty\right), \downarrow: \left(-\infty, -\frac{1}{2}\right)$;

1e. $x - ints: (-3, 0), (2, 0), y - int: (0, -6)$;

2a. $D: (-\infty, \infty), R: (-\infty, 10]$; *2b.* $g(x) = 0$ when $x = 2 \pm \sqrt{10}$;

2c. $g(x) > 0: \left(2 - \sqrt{10}, 2 + \sqrt{10}\right)$,

$g(x) < 0: \left(-\infty, 2 - \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$; *2d.* $\uparrow: (-\infty, 2), \downarrow: (2, \infty)$;

2e. $x - ints: \left(2 - \sqrt{10}, 0\right), \left(2 + \sqrt{10}, 0\right), y - int: (0, 6)$;

3a. $D: (-\infty, \infty), R: (-\infty, 0]$; *3b.* $h(x) = 0$ when $x = 2$;

3c. $h(x) > 0: NONE$, $h(x) < 0: (-\infty, 2) \cup (2, \infty)$; *3d.* $\uparrow: (-\infty, 2), \downarrow: (2, \infty)$;

3e. $x - int: (2, 0), y - int: (0, -4)$;

4a. $D: (-\infty, \infty), R: (-\infty, -2]$; *4b.* $j(x) \neq 0$;

4c. $j(x) > 0: NONE$, $j(x) < 0: (-\infty, \infty)$; *4d.* $\uparrow: (-\infty, 0), \downarrow: (0, \infty)$;

4e. $x - ints: NONE, y - int: (0, -2)$;