As shown in the previous set of notes, the vertex of a quadratic function provides information about many features of that function and its graph (increasing/decreasing intervals, line of symmetry, maximum/minimum function value, range). However when a quadratic function is expressed in polynomial form $(f(x) = ax^2 + bx + c)$, the vertex of the quadratic function is not obvious. One way to find the vertex of a quadratic function that is in polynomial form is to use the formula $x = \frac{-b}{2a}$ to find the x-coordinate of the vertex. Once you have the x-coordinate, you can find the y-coordinate by replacing x with $\frac{-b}{2a}$ in the quadratic function $\left(f\left(\frac{-b}{2a}\right)\right)$. Just like the quadratic formula that we've worked with before, the formula $x = \frac{-b}{2a}$ is also derived by completing the square (more on this later).

<u>Example 1:</u> Find the vertex of the quadratic function $f(x) = -2x^2 + 16x - 26$.

$$x = \frac{-b}{2a}$$

$$x = \frac{-(16)}{2(-2)}$$

$$x = \frac{-16}{-4}$$

$$x = 4$$

$$f(4) = -2(4)^2 + 16(4) - 26$$

$$f(4) = -32 + 64 - 26$$

$$f(4) = 6$$

$$\mathbf{Vertex}: (\mathbf{4}, \mathbf{6})$$

<u>Example 2:</u> Find the vertex of the following quadratic functions.

a. $f(x) = 5x^2 + 20x + 14$

b. $g(x) = 9x^2 - 24x + 16$

$x = \frac{-b}{2a}$ $\quad g\left(\frac{4}{3}\right) = 9\left(\frac{4}{3}\right)^2 - 24\left(\frac{4}{3}\right) + 16$

$x = \frac{-(-24)}{2(9)}$ $\quad g\left(\frac{4}{3}\right) = 9\left(\frac{16}{9}\right) - 32 + 16$

$x = \frac{24}{18}$ $\quad g\left(\frac{4}{3}\right) = 16 - 32 + 16$

$x = \frac{4}{3}$ $\quad g\left(\frac{4}{3}\right) = 0$

$\boldsymbol{V\left(\frac{4}{3}, 0\right)}$

c. $h(x) = -3x^2 - 6x - 5$

$x = \frac{-b}{2a}$ $\quad h(-1) = -3(-1)^2 - 6(-1) - 5$

$x = \frac{-(-6)}{2(-3)}$ $\quad h(-1) = -3(1) + 6 - 5$

$x = \frac{6}{-6}$ $\quad h(-1) = -3 + 6 - 5$

$x = -1$ $\quad h(-1) = -2$

$\boldsymbol{V(-1, -2)}$

d. $j(x) = -\frac{3}{4}x^2 + 9x - 34$

All the quadratic functions we've seen so far have been in polynomial form ($f(x) = ax^2 + bx + c$). However if a quadratic function is not given in polynomial form, it can be converted to polynomial form by using polynomial multiplication and combining like terms. This is what we'll do on the next example.

Example 3: Find the vertex and the zeros for each of the following quadratic functions.

$$k(x) = -\frac{1}{2}(1-x)(x-5)$$

$$k(x) = -\frac{1}{2}(x - 5 - x^2 + 5x)$$

$$k(x) = -\frac{1}{2}(-x^2 + 6x - 5)$$

$$k(x) = \frac{1}{2}x^2 - 3x + \frac{5}{2}$$

$$x = \frac{-b}{2a}$$

$$x = \frac{-(-3)}{2\left(\frac{1}{2}\right)}$$

$$x = \frac{3}{1}$$

$$x = 3$$

Remember to find the zeros of a function, simply set the function equal to zero and solve. Setting a quadratic function equal to zero and solving means you're solving a quadratic equation, so you can solve by factoring, solve by completing the square, or solve by using the quadratic formula. Review Lessons 12, 13, or 14 for more information on those methods.

$$k(x) = -\frac{1}{2}(1-x)(x-5)$$

$$0 = -\frac{1}{2}(1-x)(x-5)$$

$$0 = 1 - x \quad ; \quad x - 5 = 0$$

$$x = 1 \quad ; \quad x = 5$$

The zeros of the function $k(x)$ are $x = 1, 5$

When substituting 3 in for x to find the y-coordinate of the vertex, you can use either the original function in factored form $\left(k(x) = -\frac{1}{2}(1-x)(x-5)\right)$ or the new version of the function in polynomial form $\left(k(x) = \frac{1}{2}x^2 - 3x + \frac{5}{2}\right)$.

$$k(3) = -\frac{1}{2}(1-3)(3-5)$$

$$k(3) = -\frac{1}{2}(-2)(-2)$$

$$k(3) = 1(-2)$$

$$k(3) = -2$$

Vertex: $(3, -2)$

$$k(3) = \frac{1}{2}(3)^2 - 3(3) + \frac{5}{2}$$

$$k(3) = \frac{1}{2}(9) - 9 + \frac{5}{2}$$

$$k(3) = \frac{9}{2} - 9 + \frac{5}{2}$$

$$k(3) = -2$$

Vertex: $(3, -2)$

Example 4: Find the vertex of the following quadratic functions, and state whether the vertex is the maximum or minimum point on the parabola. **(hint: remember to use the leading coefficient of the quadratic functions to determine whether the vertex is the maximum or minimum point)**

a. $f(x) = x^2 - 12x + 21$

$x = \frac{-b}{2a}$

$x = \frac{-(-12)}{2(1)}$

$x = \frac{12}{2}$

$x = 6$

$f(6) = 6^2 - 12(6) + 21$

$f(6) = 36 - 72 + 21$

$f(6) = -15$

Vertex: $(6, -15)$

b. $m(x) = -3(x-1)(x+5)$

$m(x) = -3(x^2 + 4x - 5)$

$m(x) = -3x^2 - 12x + 15$

$x = \frac{-b}{2a}$

$x = \frac{-(-12)}{2(-3)}$

$x = \frac{12}{-6}$

$x = -2$

$m(-2) = -3(-2-1)(-2+5)$

$m(-2) = -3(-3)(3)$

$m(-2) = 27$

Vertex: $(-2, 27)$

<u>Example 5:</u> Find the vertex and the zeros for each of the following quadratic functions.

<u>(hint: remember to replace the names of the quadratic functions $(k(x)$ and $g(x))$ with zero to find the zeros of the functions)</u>

a. $k(x) = -1(x+2)(x-8)$

$k(x) = -1(x^2 - 6x - 16)$

$k(x) = -x^2 + 6x + 16$

$x = \frac{-(6)}{2(-1)}$

$x = \frac{-6}{-2}$

$x = 3$

$k(3) = -1(3+2)(3-8)$

$k(3) = -1(5)(-5)$

$k(3) = 25$

Vertex: $(3, 25)$

b. $g(x) = -3x^2 - 6x - 5$

Since $g(x)$ is already in polynomial form, I do not need to change it in any way to find the vertex, like I have to do with $k(x)$ in part a.

$x = \frac{-(-6)}{2(-3)}$

$x = \frac{6}{-6}$

$x = -1$

$g(-1) = -3(-1)^2 - 6(-1) - 5$

$g(-1) = -3(1) + 6 - 5$

$g(-1) = -3 + 6 - 5$

$g(-1) = -2$

Vertex: $(-1, -2)$

Example 6: Find the vertex and the intercepts for each of the following quadratic functions.

(hint: remember to replace the names of the quadratic functions $(f(x)$ and $m(x))$ with zero to find the x-intercepts, and then replace x with zero to find the y-intercepts)

a. $f(x) = 2x^2 - 8x + 3$

$x = \frac{-(-8)}{2(2)}$

$x = \frac{8}{4}$

$x = 2$

$f(2) = 2(2)^2 - 8(2) + 3$

$f(2) = 2(4) - 16 + 3$

$f(2) = 8 - 16 + 3$

$f(2) = -5$

Vertex: $(2, -5)$

b. $m(x) = 5(x + 4)(x - 2)$

$m(x) = 5(x^2 + 2x - 8)$

$m(x) = 5x^2 + 10x - 40$

$x = \frac{-(10)}{2(5)}$

$x = \frac{-10}{10}$

$x = -1$

$m(-1) = 5(-1 + 4)(-1 - 2)$

$m(-1) = 5(3)(-3)$

$m(-1) = -45$

Vertex: $(-1, -45)$

Example 7: Find the vertex and the range for each of the following quadratic functions.

a. $k(x) = -2(x-3)(x+7)$

$k(x) = -2(x^2 + 4x - 21)$

$k(x) = -2x^2 - 8x + 42$

$x = \frac{-(-8)}{2(-2)}$

$x = \frac{8}{-4}$

$x = -2$

$k(-2) = -2(-2-3)(-2+7)$

$k(-2) = -2(-5)(5)$

$k(-2) = 50$

Vertex: $(-2, 50)$

b. $g(x) = x^2 + 2x - 8$

$x = \frac{-2}{2(1)}$

$x = \frac{-2}{2}$

$x = -1$

$g(-1) = (-1)^2 + 2(-1) - 8$

$g(-1) = 1 - 2 - 8$

$g(-1) = -9$

Vertex: $(-1, -9)$

In order to find the range of each function, I simply need to use the y-coordinate of each vertex and determine whether that is the largest output or the smallest output. To do so, I use the leading coefficient of each function to determine whether the graph is opening up or down.

The leading coefficient of $k(x)$ is -2, so the graph of $k(x)$ is opening down. That means 50 is the largest output

Range: $(-\infty, 50]$

The leading coefficient of $g(x)$ is 1, so the graph of $g(x)$ is opening up. That means -9 is the smallest output

Range: $[-9, \infty)$

Example 8: Find the vertex and the increasing/decreasing intervals for each of the following quadratic functions.

(hint: remember to use the leading coefficient a for each quadratic function and the x-coordinate of each vertex to find the increasing/decreasing intervals)

a. $f(x) = -\frac{1}{2}(3-x)(x+2)$

$f(x) = -\frac{1}{2}(-x^2 + x + 6)$

$f(x) = \frac{1}{2}x^2 - \frac{1}{2}x - 3$

$x = \frac{-\left(-\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)}$

$x = \frac{\frac{1}{2}}{1}$

$x = \frac{1}{2}$

$f\left(\frac{1}{2}\right) = -\frac{1}{2}\left(3 - \frac{1}{2}\right)\left(\frac{1}{2} + 2\right)$

$f(3) = -\frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{5}{2}\right)$

$f(3) = -\frac{25}{8}$

Vertex: $\left(\frac{1}{2}, -\frac{25}{8}\right)$

b. $h(x) = \frac{2}{5}x^2 - \frac{12}{5}x + \frac{23}{5}$

$x = \frac{-\left(-\frac{12}{5}\right)}{2\left(\frac{2}{5}\right)}$

$x = \frac{\frac{12}{5}}{\frac{4}{5}}$

$x = \frac{12}{4}$

$x = 3$

$h(3) = \frac{2}{5}(3)^2 - \frac{12}{5}(3) + \frac{23}{5}$

$h(3) = \frac{2}{5}(9) - \frac{36}{5} + \frac{23}{5}$

$h(3) = \frac{18}{5} - \frac{36}{5} + \frac{23}{5}$

$h(3) = 1$

Vertex: $(3, 1)$

Answers to Examples:

2a. $V(-2,-6)$; *2b.* $V\left(\frac{4}{3},0\right)$; *2c.* $V(-1,-2)$; *2d.* $V(6,-7)$;

3a. $V(3,-2)$, zeros: $x=1,5$;

4a. $V(6,-15)$, MIN ; *4b.* $V(-2,27)$, MAX ;

5a. $V(3,25), k(x)=0$ when $x=-2,8$;

5b. $V(-1,-2)$, NONE $(g(x)\neq 0)$;

6a. $V(2,-5), x-$ intercepts: $\left(2+\frac{\sqrt{10}}{2},0\right),\left(2-\frac{\sqrt{10}}{2},0\right)$,
$y-$ intercepts: $(0,3)$;

6b. $V(-1,-45), x-$ intercepts: $(-4,0),(2,0)$,
$y-$ intercepts: $(0,-40)$;

7a. $V(-2,50)$, Range: $(-\infty,50]$;

7b. $V(-1,-9)$, Range: $[-9,\infty)$;

8a. $V\left(\frac{1}{2},-\frac{25}{8}\right)$, Increasing: $\left(\frac{1}{2},\infty\right)$, Decreasing: $\left(-\infty,\frac{1}{2}\right)$;

8b. $V(3,1)$, Increasing: $(3,\infty)$, Decreasing: $(-\infty,3)$;