Logarithmic Functions:

- the inverse of an exponential function
  - the function that undoes an exponential function
  - if $f(x) = a^x$, then $f^{-1}(x) = \log_a(x)$

$f(x) = a^x$	$f^{-1}(x) = \log_a(x)$
Domain: $(-\infty, \infty)$	Domain: $(0, \infty)$
Range: $(0, \infty)$	Range: $(-\infty, \infty)$
Any real number can be the input of an exponential function $f(x) = a^x$, but only positive outputs are possible because a positive number to the power of any real number is positive.	Only positive real numbers can be the argument of a logarithmic function $g(x) = \log_a(x)$, but any real number is possible for the output because the output of a logarithmic function is the same as the input of its inverse (an exponential function), an exponent.

When $f(x) = \log_a(x)$, the argument (the quantity in parentheses) is a power of the base a and the output is the exponent which produces that power. The argument must be a positive number, and the base a must be a positive number other than 1.

A logarithm is an exponent; a logarithm represents the exponent needed to change a base into a power of that base (the argument). When finding function values for a logarithmic function, think about what exponent is needed to make the base equal to the argument. For a function like $f(x) = \log_3(x)$, we need to determine what exponent make 3 equal to x.

$$f(9) = \log_3(9) = 2 \text{ because } 3^2 = 9$$

$$f(1) = \log_3(1) = 0 \text{ because } 3^0 = 1$$

$$f\left(\frac{1}{3}\right) = \log_3\left(\frac{1}{3}\right) = -1 \text{ because } 3^{-1} = \frac{1}{3}$$

Common logarithm:

This is similar to how a variable with no exponent has an understood power of 1 ($x = x^1$) or how a radical with no index has an understood root of 2 ($\sqrt{x} = \sqrt[2]{x}$)

- $f(x) = \log(x)$
  - when no base is denoted, it is understood to be base 10
- $f(x) = \log(x)$ is equivalent to $f(x) = \log_{10}(x)$

Just like any other logarithms, common logs are exponents. A common log is the exponent needed to make 10 become a power of 10

$$\log(100) = 2 \text{ because } 10^2 = 100$$

$$\log\left(\frac{1}{10{,}000}\right) = -4 \text{ because } 10^{-4} = \frac{1}{10{,}000}$$

Keep in mind that logarithms are simply exponents; when working with logarithms, ask yourself what exponent makes the base equal to the argument.

Example 1: Given the logarithmic function $f(x) = \log(x)$, find the following function values. If no function value exists for a particular input, write DNE. **You may use your calculator to find these function values.**

a. $f(10) = \log(10)$

$\log_{10}(10) = \mathbf{1}$ because $10^{\mathbf{1}} = 10$

b. $f(100) = \log(100)$

$\log_{10}(100) = \mathbf{2}$ because $10^{\mathbf{2}} = 100$

c. $f(1000) = \log(1000)$

$\log_{10}(1000) = \mathbf{3}$ because $10^{\mathbf{3}} = 1000$

d. $f\left(\frac{1}{10}\right) = \log\left(\frac{1}{10}\right)$

e. $f(0.01) = \log(0.01)$

f. $f\left(\frac{1}{1000}\right) = \log\left(\frac{1}{1000}\right)$

g. $f(-10) = \log(-10)$

h. $f(1) = \log(1)$

i. $f(0) = \log(0)$

j. $f(0.0001) + f(1{,}000{,}000) - f\left(\sqrt[3]{100}\right) + f\left(\sqrt{0.1}\right)$

$$\log(0.0001) + \log(1{,}000{,}000) - \log\left(\sqrt[3]{100}\right) + \log\left(\sqrt{0.1}\right)$$

$$-4 + 6 - \frac{2}{3} + -\frac{1}{2} = \mathbf{\frac{13}{6}}$$

Natural logarithm:

- $g(x) = \ln(x)$
  - this is a logarithm with a base of e
- $g(x) = \ln(x)$ is equivalent to $g(x) = \log_e(x)$
  - ln means log base e, so if you write ln , there is no need to include a base
  - also, do not write x (the input) as the base
- when you see $g(x) = \ln(x)$, feel free to re-write the function as $g(x) = \log_e(x)$

Example 2: Given the logarithmic function $g(x) = \ln(x)$, find the following function values. If no function value exists for a particular input, write DNE. **You may use your calculator to find these function values.**

a. $g(e) = \ln(e)$

$\log_e(e) = \mathbf{1}$ because $e^{\mathbf{1}} = e$

b. $g\left(\frac{1}{e}\right) = \ln\left(\frac{1}{e}\right)$

$\log_e\left(\frac{1}{e}\right) = \mathbf{-1}$ because $e^{-1} = \frac{1}{e}$

c. $g\left(-\frac{1}{e}\right) = \ln\left(-\frac{1}{e}\right)$

$\log_e\left(-\frac{1}{e}\right)$ DNE because $-\frac{1}{e} < 0$

d. $g(e^2) = \ln(e^2)$

e. $g\left(\frac{1}{e^2}\right) = \ln\left(\frac{1}{e^2}\right)$

f. $g(-1) = \ln(-1)$

g. $g(1) = \ln(1)$

h. $g(0) = \ln(0)$

i. $g(e^5) = \ln(e^5)$

j. $g(\sqrt{e}) \quad - \quad g\left(\frac{1}{\sqrt{e}}\right) + g(e^{\pi})$

$\ln(\sqrt{e}) - \ln\left(\frac{1}{\sqrt{e}}\right) + \ln(e^{\pi})$

$\frac{1}{2} \quad - \left(-\frac{1}{2}\right) + \quad \pi \quad = \mathbf{1 + \pi}$

As mentioned on page 1 of these notes, the domain of a logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$. The two logarithmic functions we worked with today (common logs and natural logs) are no different:

$f(x) = \log(x)$ $\qquad$ $g(x) = \ln(x)$

Domain: $(\mathbf{0}, \infty)$ $\qquad$ **Domain**: $(\mathbf{0}, \infty)$

The domains of $f(x) = \log(x)$ and $g(x) = \ln(x)$ are limited to positive numbers only because the input x is the argument of the logarithm, and the argument must be positive because it is either a power of 10 or a power of e.

Range: $(-\infty, \infty)$ $\qquad$ **Range**: $(-\infty, \infty)$

The ranges of $f(x) = \log(x)$ and $g(x) = \ln(x)$ are all real numbers because the output of a logarithmic function is an exponent, and an exponent can be any real number.

Inputs	**Outputs**
x	$f(x) = \log(x)$
$\frac{1}{100}$	$\log\left(\frac{1}{100}\right) = -2$
$\frac{1}{10}$	$\log\left(\frac{1}{10}\right) = -1$
1	$\log(1) = 0$
10	$\log(10) = 1$
100	$\log(100) = 2$

Inputs	**Outputs**
x	$f(x) = \ln(x)$
$\frac{1}{e^2}$	$\ln\left(\frac{1}{e^2}\right) = -2$
$\frac{1}{e}$	$\ln\left(\frac{1}{e}\right) = -1$
1	$\ln(1) = 0$
e	$\ln(e) = 1$
e^2	$\ln(e^2) = 2$

Keep in mind that just like any other function, logarithmic functions can be transformed, and transforming a log function could change its domain.

Answers to Examples:

1a. 1 ; *1b.* 2 ; *1c.* 3; *1d.* -1 ; *1e.* -2 ; *1f.* -3 ; *1g.* DNE ;
1h. 0 ; *1i.* DNE ; *1j.* $\frac{5}{6}$;
2a. 1 ; *2b.* -1 ; *2c.* DNE ; *2d.* 2 ; *2e.* -2 ; *2f.* DNE ; *2g.* 0 ;
2h. DNE ; *2i.* 5 ; *2j.* $1+\pi$;