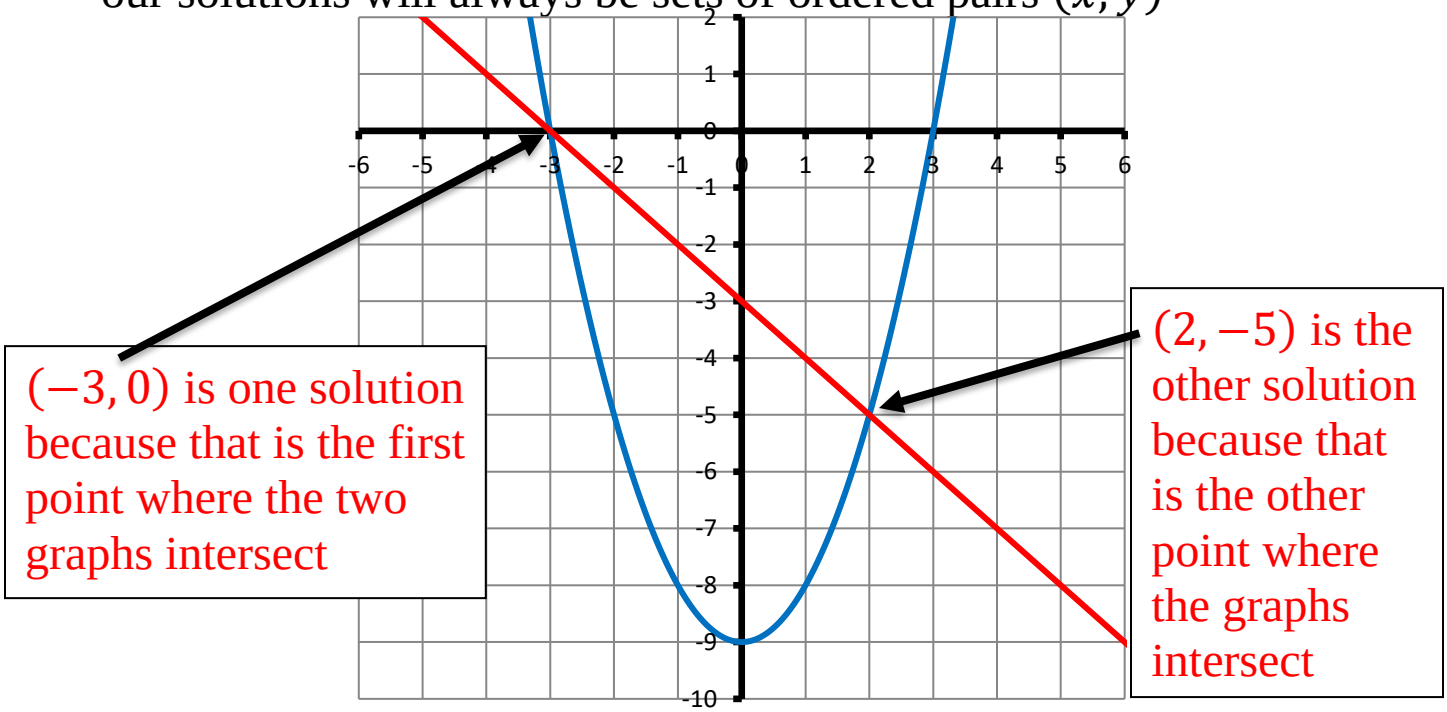
## System of equations:

- two or more equations containing common variables
  - Example: $\begin{cases} x^2 - y = 9 \\ x + y = -3 \end{cases}$
  - just like in the example shown, the variables will not always be of the same degree; be aware of this when making substitutions
- the solution set of a system of equations are the sets of ordered pairs that make each equation true
  - the solution set of the system given above is $(-3,0), (2,-5)$

$$\begin{cases} (-3)^2 - 0 = 9 \\ -3 + 0 = -3 \end{cases} \qquad \begin{cases} (2)^2 - (-5) = 9 \\ 2 + (-5) = -3 \end{cases}$$

- we will only cover systems with two equation and two variables, so our solutions will always be sets of ordered pairs $(x, y)$




When a system of equations is graphed, the solutions are the points where the graphs of the equations intersect. If the graphs never intersect, such as parallel lines, the system has no solution because there are no intersection points. If the graphs are the same, the system has infinitely many solutions because the graphs intersect at every point. We will not cover how to solve systems of equations graphically in this class, but thinking about the solutions from a graphical standpoint can help to make more sense of them.

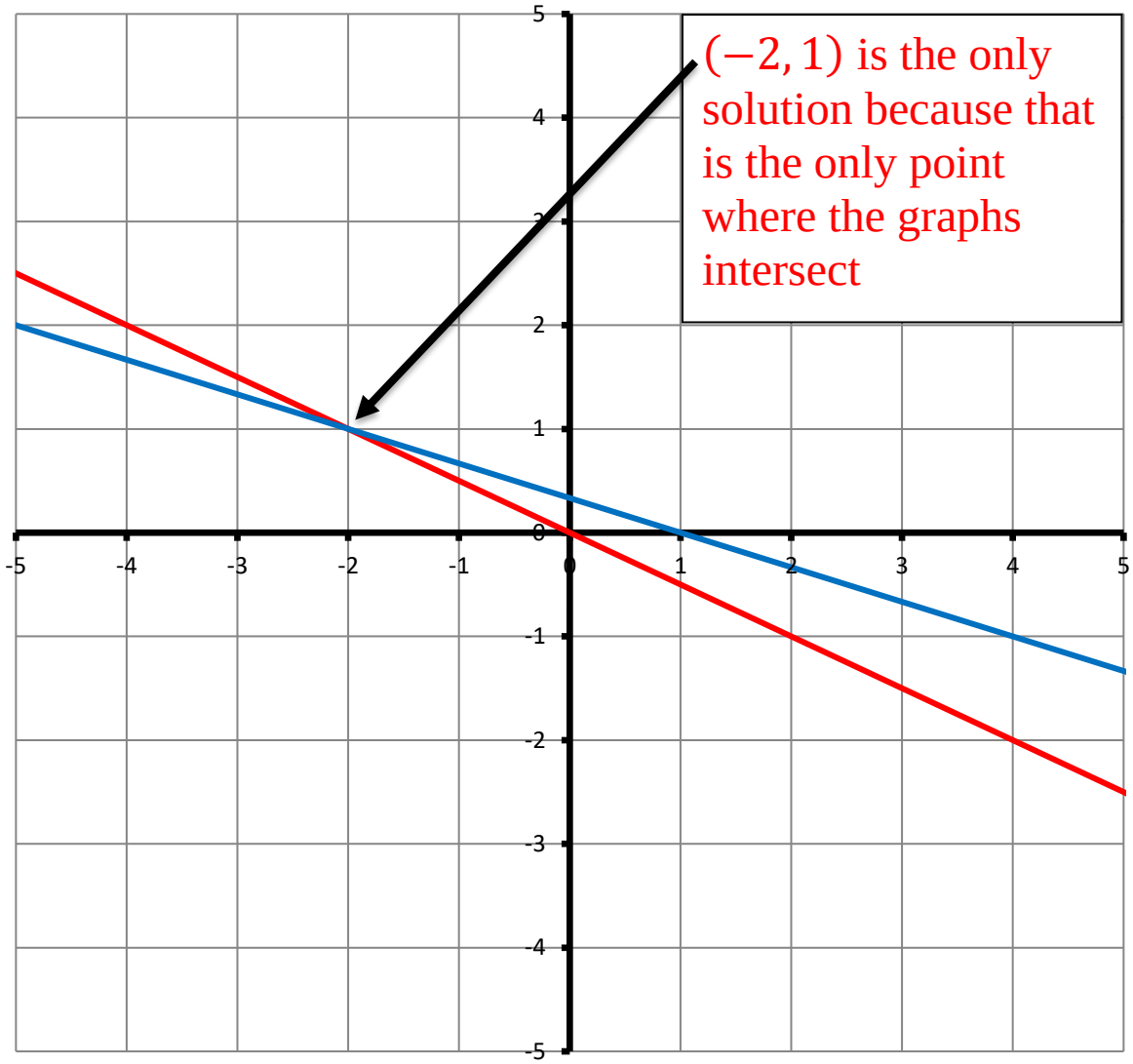
The two methods we will use to solve systems are substitution and elimination. We will cover substitution in this lesson and elimination in the next.

**Method of Substitution:**

1. solve one equation for one variable
   a. **IT DOESN'T MATTER WHICH EQUATION YOU CHOOSE OR WHICH VARIABLE YOU CHOOSE TO SOLVE**
2. substitute the solution from step 1 into the other equation
3. solve the new equation from step 2
4. back substitute to solve the equation from step 1

**Example 1:** Use the method of substitution to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$$\begin{cases} x + 3y = 1 \\ \frac{x}{y} = -2 \end{cases}$$




The graphs of the equations above give us a visual representation of what the solutions of our system means. We can see that the graphs of the two equations intersect at the point $(-2, 1)$, which is the solution of the system.

Keep in mind that since the original second equation was $\frac{x}{y} = -2$, an answer of $y = 0$ would not have been valid since the denominator of a fraction can never equal zero. Be aware of systems that contain fractions, square roots, and/or logarithms, since those equations will have restrictions. We'll see this again in upcoming examples, as well as in the homework.

**Example 2:** Use the method of substitution to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$\begin{cases} x - y = -4 \\ x^2 - y = -2 \end{cases}$ ; on this problem, I would solve the first equation for $y$, then I would substitute my answer into the second equation.

$$x - y = -4$$

$$x + 4 = y$$

Replacing $y$ with $x + 4$ in the second equation results in the following new equation:

$$x^2 - (x + 4) = -2$$

$$x^2 - x - 4 = -2$$

$$x^2 - x - 2 = 0$$

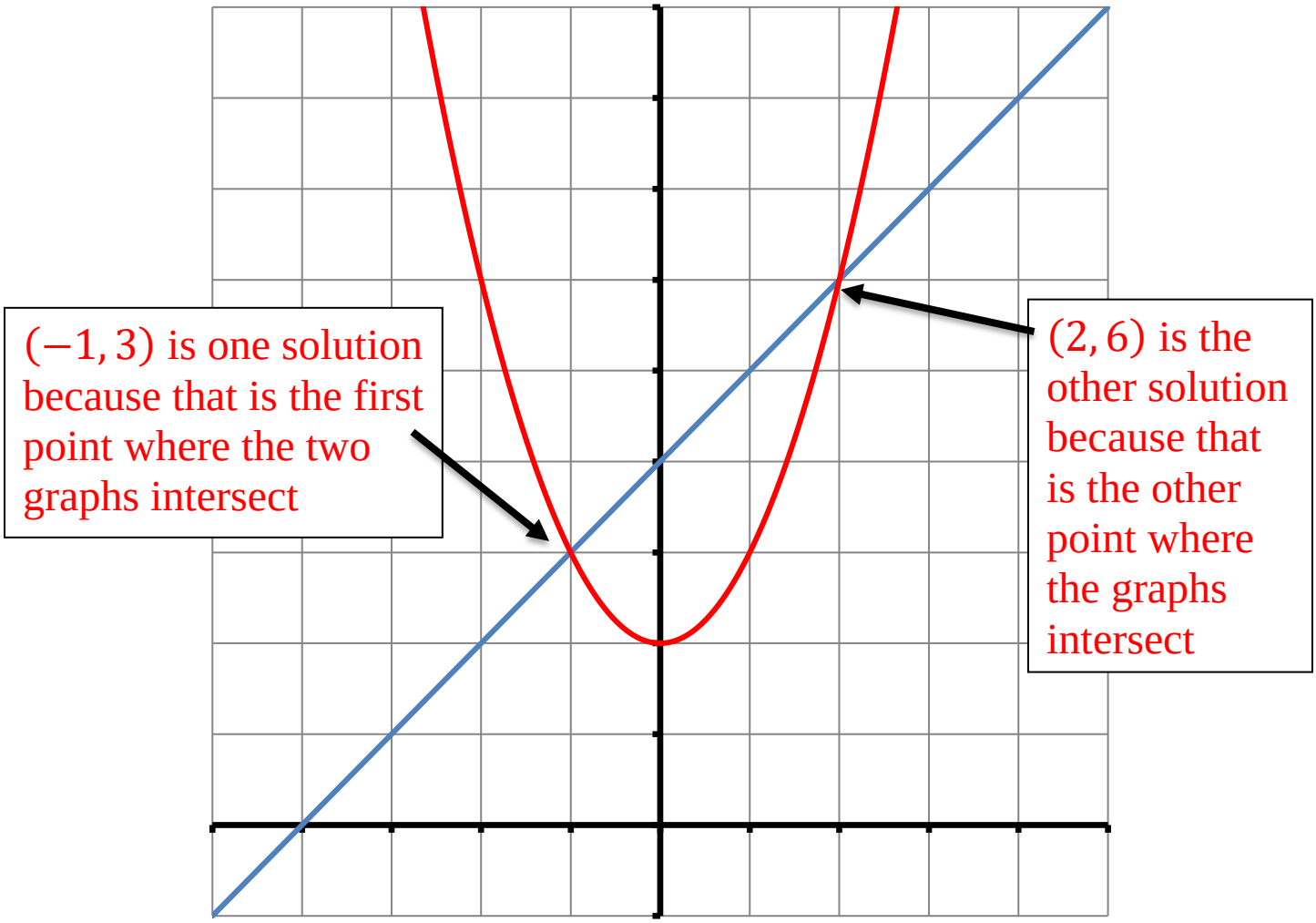
$$(x - 2)(x + 1) = 0$$

$$x = 2 \quad ; \quad x = -1$$

Now I can back substitute to solve for $y$ in the equation $x + 4 = y$.

| | |
|---|---|
| $x = 2$ | $x = -1$ |
| $y = x + 4$ | $y = x + 4$ |
| $y = 2 + 4$ | $y = -1 + 4$ |
| $y = 6$ | $y = 3$ |

When $x = 2, y = 6$, so I get the ordered pair $(2, 6)$. And when $x = -1$, $y = 3$, so I get the ordered pair $(-1, 3)$. So the solution for the system of equations $\begin{cases} x - y = -4 \\ x^2 - y = -2 \end{cases}$ are $\mathbf{(2, 6)}$ and $\mathbf{(-1, 3)}$.



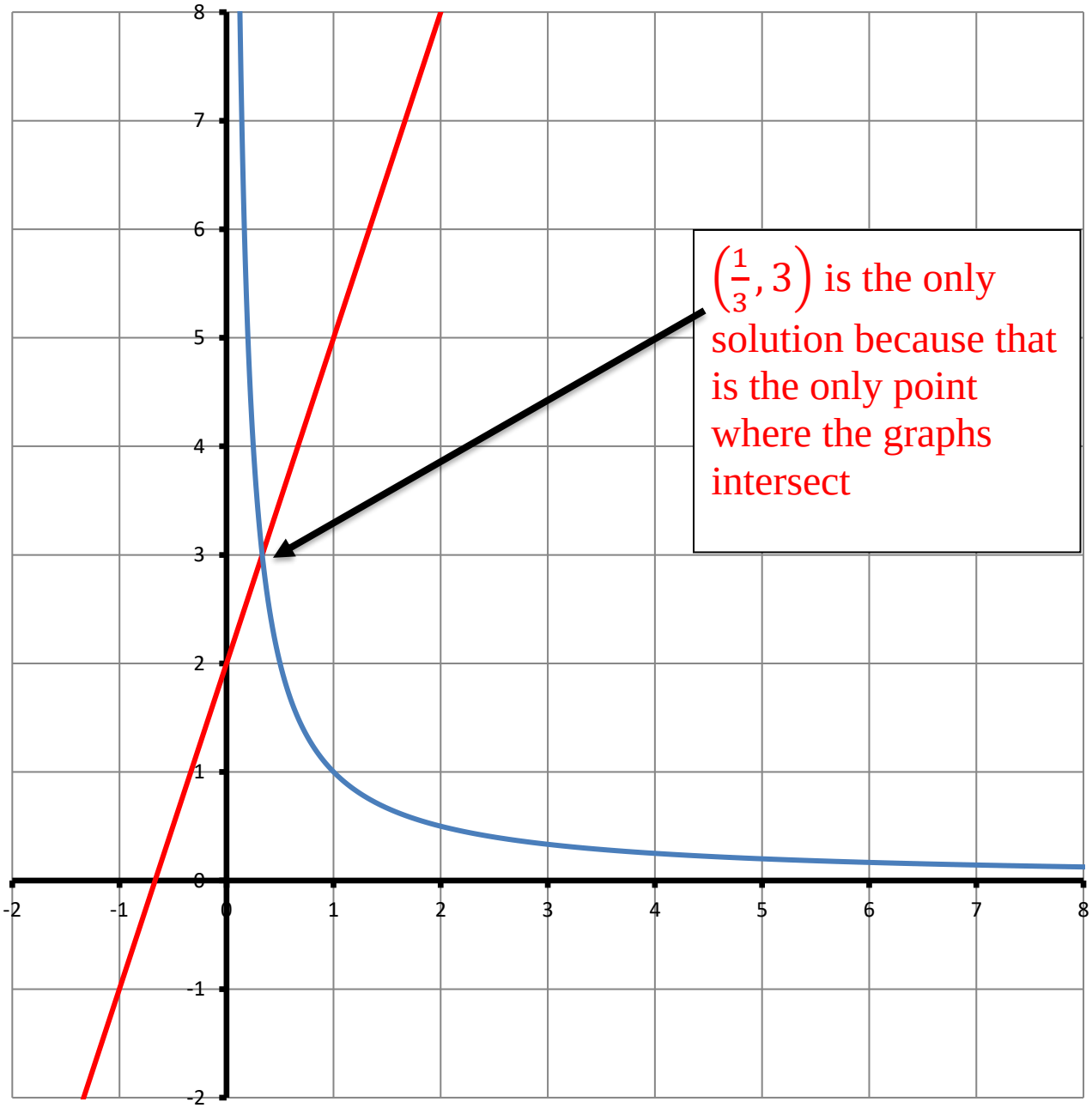

You can verify whether your solution(s) is/are correct by plugging the ordered pairs back into the original equations. If the ordered pairs in your solution set make both of the original equations true, your answers are correct.

In Example 2, checking the solutions was optional because both equations ($x - y = -4$ and $x^2 - y = -2$) are defined for all values of $x$ (there were no restrictions because of fractions, square roots, or logarithms). In Example 3, we will have a system of equations which contains logarithms, so we will be required to check our solutions to verify they result in positive arguments.

**<u>Example 3:</u>** Use the method of substitution to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

**<u>BE SURE TO CHECK YOUR ANSWERS SINCE THIS SYSTEM CONTAINS AN EQUATION WITH LOGARITHMS.</u>**

$$\begin{cases} \log(x) + \log(y) = 0 \\ 3x - y = -2 \end{cases}$$



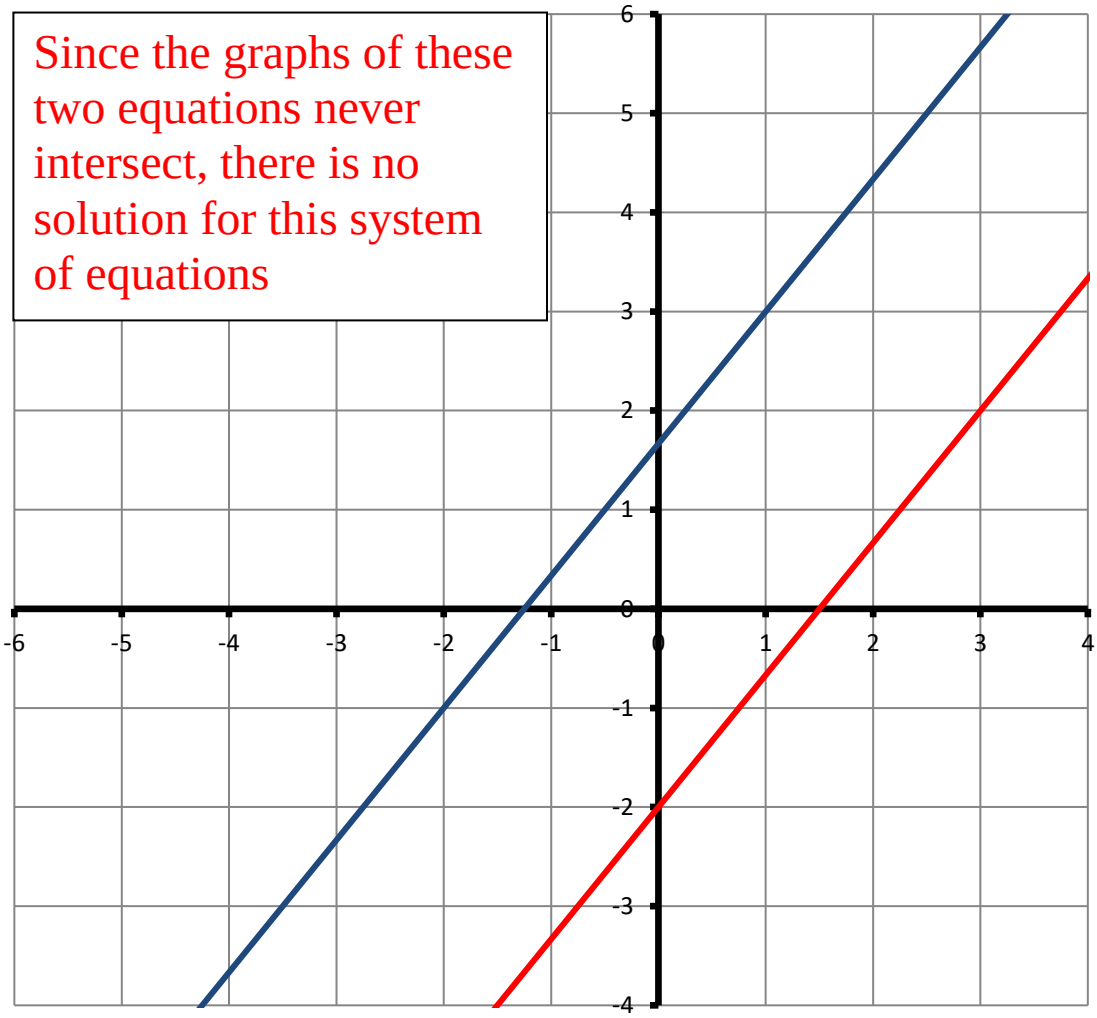

We can see from the graphs that the two equations only intersect at the point $\left(\frac{1}{3}, 3\right)$. The graph of $\log(x) + \log(y) = 0$ only lies in Quadrant I, so any ordered pairs that lie outside Quadrant I (such as the point $(-1, -1)$) cannot be solutions since $x$ and $y$ must both be positive.

Converting logarithmic equations to exponential form (Lesson 33) and converting exponential equations to logarithmic form (Lesson 34) come up a lot in Lessons 35 & 36 when solving systems of equations. If you need to review how to convert from one form to the other, be sure to do so.

**<u>Example 4:</u>** Use the method of substitution to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

(hint: getting rid of the fractions will make it more difficult to make a substitution, so make a substitution first)

$$\begin{cases} x = \frac{3}{4}y + \frac{3}{2} \\ -\frac{4}{3}x + y = \frac{5}{3} \end{cases}$$



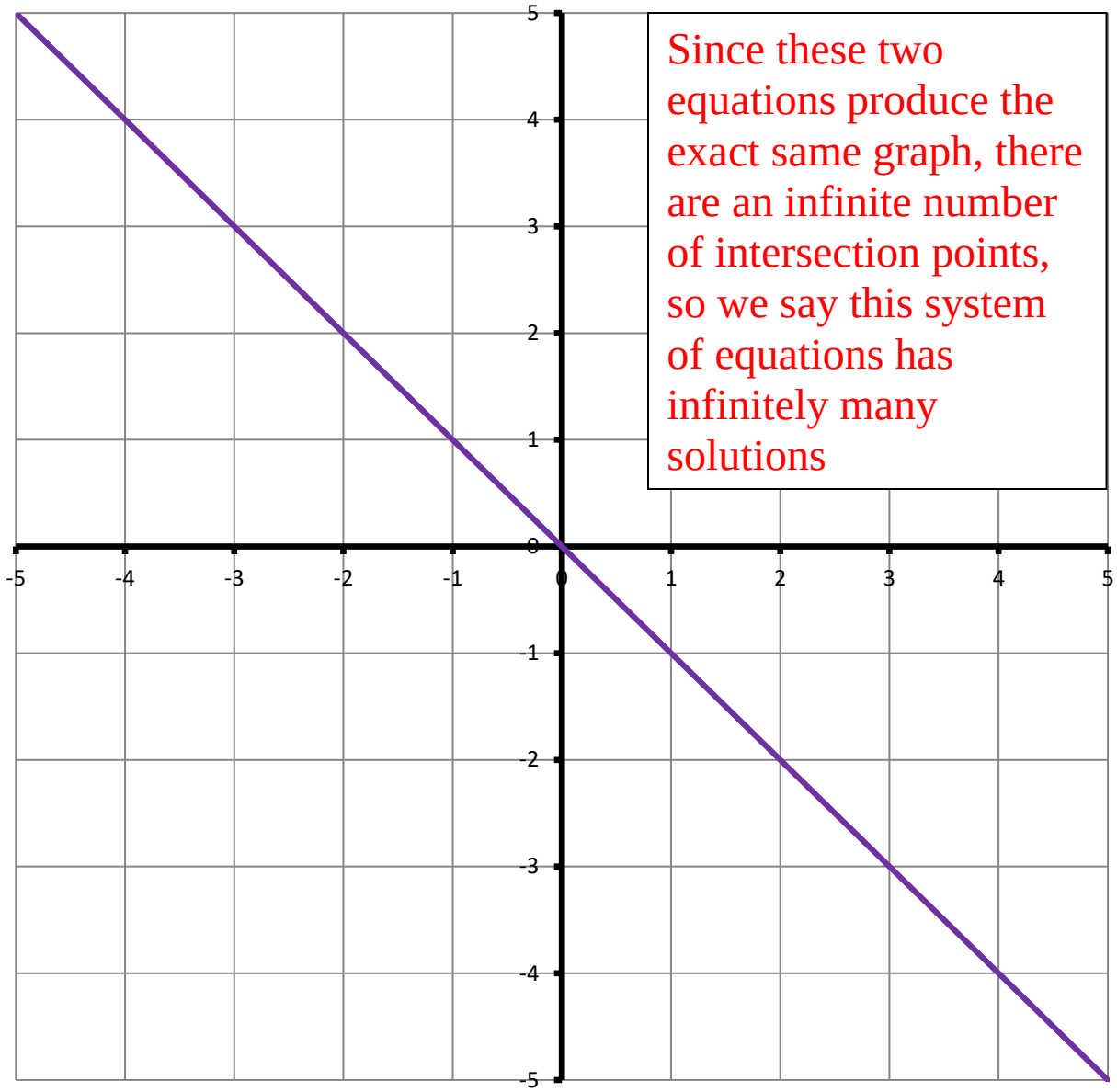

A false statement, such as $-2 = \frac{5}{3}$, is always an indication that the system has no solutions (NS) and that the graphs never intersect.

There are two different scenarios in which we can determine that a particular system of equations has no solution. One is to end up with an equation that is not true, such $-2 = \frac{5}{3}$. The other is to end up with an expression that is undefined, such as a logarithm with a negative argument or a fraction with a denominator of zero. Both of these scenarios will occur either in this lesson or the next when solving systems of equations.

An example of a system that will produce an expression that is undefined is $\begin{cases} \log_2(y-1) = x \\ y = -2^{x-3} \end{cases}$. Solving the system results in a $y$ value of $\frac{1}{9}$, which is not possible since $y - 1$ is the argument of the logarithm. If you only have one solution, and it is not valid, you're left with no solutions.

**<u>Example 5:</u>** Use the method of substitution to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$$\begin{cases} 10^{x+y} - 1 = 0 \\ x = -y \end{cases}$$




A true statement, such as $0 = 0$, is always an indication that the system has infinitely many solutions (IMS) and that the graphs are the same. A false statement, such as 0 being equal to any other number besides 0, is always an indication that the system has no solutions (NS) and that the graphs never intersect.

It is not obvious that the equations $10^{x+y} - 1 = 0$ and $x = -y$ are in fact the same, but if you manipulate the equation $10^{x+y} - 1 = 0$ by isolating the exponential expression and then converting to log form, you will end up with $0 = x + y$, which is the same as $-y = x$.

**Example 6:** Use the method of substitution to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$\begin{cases} y - e^{-x} = -1 \\ x = \ln(1-y) \end{cases}$ ; on this problem, I would solve the first equation for $x$, then I would substitute my answer into the second equation.

$$y - e^{-x} = -1$$

$$y + 1 = e^{-x}$$

$$\log_e(y+1) = -x$$

$$-\ln(y+1) = x$$

Replacing $x$ with $-\ln(y+1)$ in the second equation results in the following new equation:

$$-\ln(y+1) = \ln(1-y)$$

$$0 = \ln(1-y) + \ln(y+1)$$

$$0 = \ln\big((1-y)(y+1)\big)$$

$$0 = \ln(y + 1 - y^2 - y)$$

$$0 = \log_e(1-y^2)$$

$$e^0 = 1 - y^2$$

$$1 = 1 - y^2$$

$$y^2 = 0$$

$$y = 0$$

Now I can back substitute to solve for $x$.

$$x = -\ln(1-y)$$

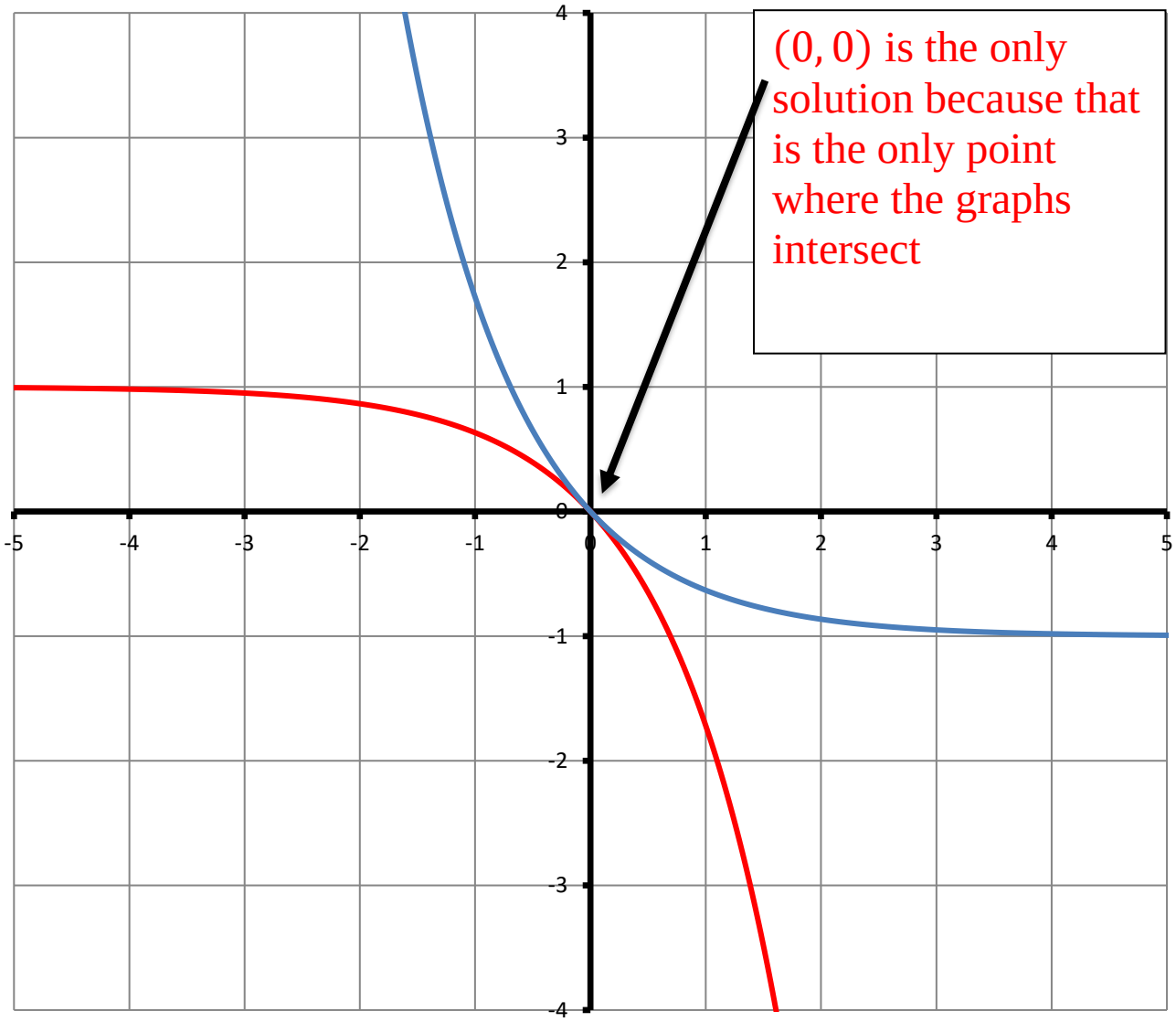
$$x = -\ln(1-0)$$

$$x = -\log_e(1)$$

$$x = -0$$

$$x = 0$$

When $y = 0, x = 0$, so I get the ordered pair $(0,0)$. So the solution for the system of equations $\begin{cases} y - e^{-x} = -1 \\ x = \ln(1-y) \end{cases}$ is $\mathbf{(0,0)}$.




There will be a lot of review when working with systems of equations, because there are numerous concepts that we've already covered that might be needed in order to solve a particular system (solving quadratic equations, converting exponential equations to logs, converting logarithmic equations to exponentials, using the Properties of Logarithms, etc.). Reviewing those concepts now will be good practice since the Final Exam will be cumulative.

**Example 7:** Solve each system and write your answers as an ordered pair. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

a. $\begin{cases} \log_x(2y) = 3 \\ \log_x(4y) = 2 \end{cases}$ ; Before solving for either variable and attempting to make a substitution, I will first convert both equations to exponential form.

$$\begin{cases} \log_x(2y) = 3 \\ \log_x(4y) = 2 \end{cases} \quad \rightarrow \quad \begin{cases} 2y = x^3 \\ 4y = x^2 \end{cases}$$

Now that both equations are in exponential form, I will solve the first equation for $y$.

$$2y = x^3$$

$$y = \frac{1}{2}x^3$$

Now that I know $y = \frac{1}{2}x^3$, I can substitute that into the second equation.

$$4y = x^2$$

$$4\left(\frac{1}{2}x^3\right) = x^2$$

$$2x^3 = x^2$$

$$2x^3 - x^2 = 0$$

$$x^2(2x - 1) = 0$$

$$x^2 = 0 \quad ; \quad 2x - 1 = 0$$

$$x = 0 \quad ; \quad x = \frac{1}{2}$$

$$x = \frac{1}{2}$$

$$y = \frac{1}{2}x^3$$

$$y = \frac{1}{2}\left(\frac{1}{2}\right)^3$$

$$y = \frac{1}{16}$$

$$\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{16}}\right)$$

We have two values for $x$, 0 and $\frac{1}{2}$, but only one is valid. Since $x$ is the base of a logarithm, it must be a positive number other than 1. So I will only use $x = \frac{1}{2}$ when I back substitute to solve for $y$.

b. $\begin{cases} e^{x+y} = 6 \\ e^{x-y} = -7 \end{cases}$

Once again before solving for either variable and attempting to make a substitution, I will first convert both equations, this time to logarithmic form.

$$\begin{cases} e^{x+y} = 6 \\ e^{x-y} = -7 \end{cases} \quad \rightarrow \quad \begin{cases} \log_e(6) = x + y \\ \log_e(-7) = x - y \end{cases}$$

At this point I notice an issue; the second equation is $\log_e(-7) = x - y$. The argument of a logarithm must **ALWAYS** be positive, because the argument of a logarithm is a power of the base. And in this case, every power of base $e$ is a positive only. So the second equation is not true. In fact it wasn't true to begin with, because $e^{x-y}$ cannot be equal to a negative value, such as $-7$. So because we have an equation that is not true, we say this system will have **NO SOLUTION**.

# NS

c. $\begin{cases} \log(xy) = 7 \\ \log(x^4) + \log(y^4) = 28 \end{cases}$

d. $\begin{cases} y = \ln(x) \\ x = 2^y \end{cases}$

*Answers to Examples:*

*7a.* $\left(\frac{1}{2}, \frac{1}{16}\right)$ *; 7b.* NS *; 7c.* IMS *; 7d.* $(1, 0)$ *;*