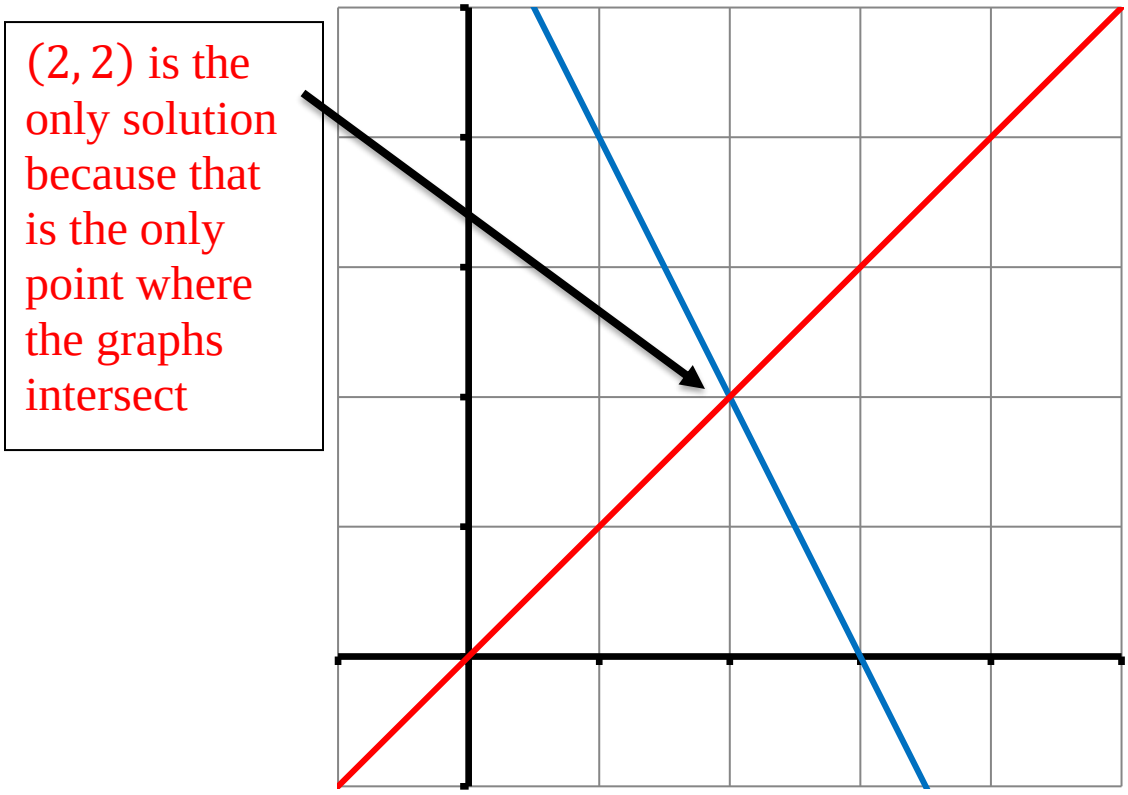
The two methods we will use to solve systems are substitution and elimination. Substitution was covered in the last lesson and elimination is covered in this lesson.

<u>Method of Elimination:</u>

1. multiply at least one equation by a nonzero constant so the coefficients for one variable will be opposites (same absolute value)
2. add the equations so the variable with the opposite coefficients will be eliminated
3. take the result from step 2 and solve for the remaining variable
4. take the solution from step 3 and back substitute to any of the equations to solve for the remaining variable
    a. **<u>YOU CAN BACK SUBSTITUTE TO ANY OF THE PREVIOUS EQUATIONS</u>**

<u>Example 1:</u> Use the method of elimination to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$$\begin{cases} 2x + y = 6 \\ -x + y = 0 \end{cases}$$

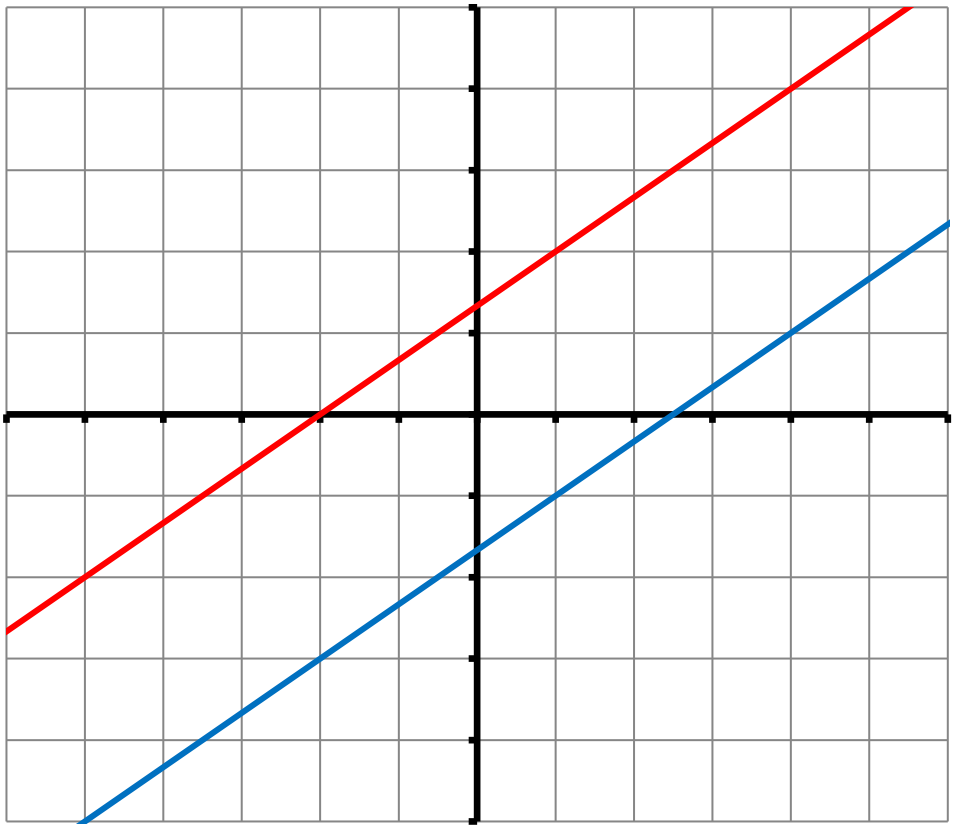


Keep in mind the #1 rule of algebra still applies (whatever you do to one side of an equation, you must do to the other side). However, you do not need to do the same thing to each equation. In other words you could multiply one equation by one non-zero constant, and multiply the other equation by a completely different non-zero constant (or not multiply the other equation at all). You must do the same thing to both sides of an equation, but you do **NOT** need to do the same thing to two different equations.

<u>Example 2:</u> Use the method of elimination to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$$\begin{cases} 2x - 3y = 5 \\ -6x + 9y = 12 \end{cases}$$

Since the graphs of these two equations never intersect, there is no solution for this system of equations

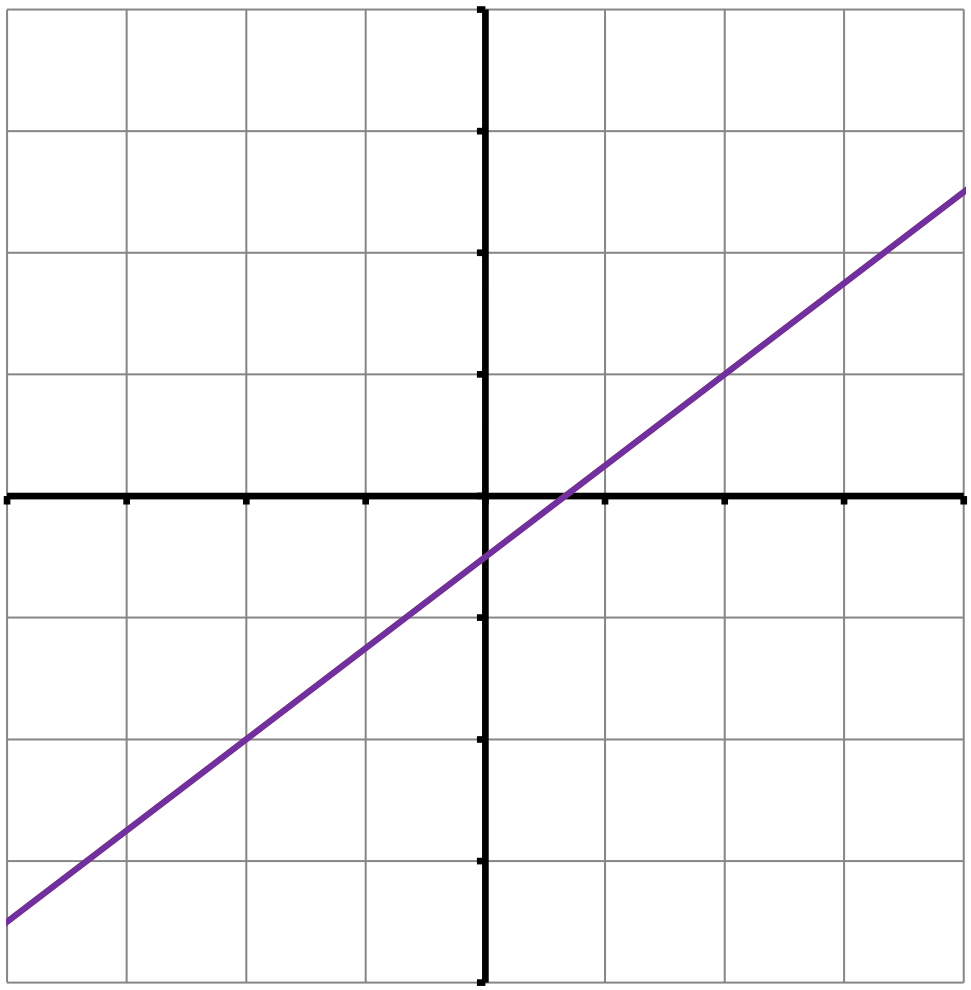


As we saw in the previous lesson, a false statement (such as $0 = 27$) is an indication that we will have no solution (NS). If we had taken the first equation of the system $\begin{cases} 2x - 3y = 5 \\ -6x + 9y = 12 \end{cases}$ and multiplied by -3, we would have had the system $\begin{cases} -6x + 9y = -15 \\ -6x + 9y = 12 \end{cases}$ which is saying that the same expression $-6x + 9y$ is equal to two different values. This is another way of showing that this system has no solution. Graphically we can see that the graphs of the equations will never intersect because they are parallel lines, so this is one more way of showing there is no solution for this system.

<u>Example 3:</u> Use the method of elimination to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$$\begin{cases} 3x - 4y &= 2 \\ -6x + 8y &= -4 \end{cases}$$

Since these two equations produce the exact same graph, there are an infinite number of intersection points, so we say this system of equations has infinitely many solutions



As we saw in the previous lesson, a true statement (such as $0 = 0$) is an indication that we will have infinitely many solutions (IMS). If we had taken the first equation of the system $\begin{cases} 3x - 4y = 2 \\ -6x + 8y = -4 \end{cases}$ and multiplied by -2, we would have had the system $\begin{cases} -6x + 8y = -4 \\ -6x + 8y = -4 \end{cases}$ in which both equations are the same. So this is another way of showing that this system has infinitely many solutions. And graphically that means that both equations will produce the same graph, so they'll intersect at every point that they pass through.

Keep in mind that infinitely many solutions does **<u>NOT</u>** mean that every ordered pair in the coordinate system is a solution; it means that every ordered pair that lies on graph of the equations is a solution.

Example 4: Use the method of elimination to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$$\begin{cases} e^x + e^y = 10 \\ e^x - e^y = -8 \end{cases}$$

To solve this system of equations using elimination I will simply add the two equations together so that e^y and $-e^y$ cancel each other out:

$$2(e^x) = 2$$

$$e^x = 1$$

$$x = \log_e(1)$$

Remember that a logarithm is an exponent; $\log_e(1)$ represents the exponent that makes the base e equal to the argument 1.

$$x = 0$$

Now that I have the value of x, I'll back substitute to find the value of y:

$$e^x + e^y = 10$$

$$e^0 + e^y = 10$$

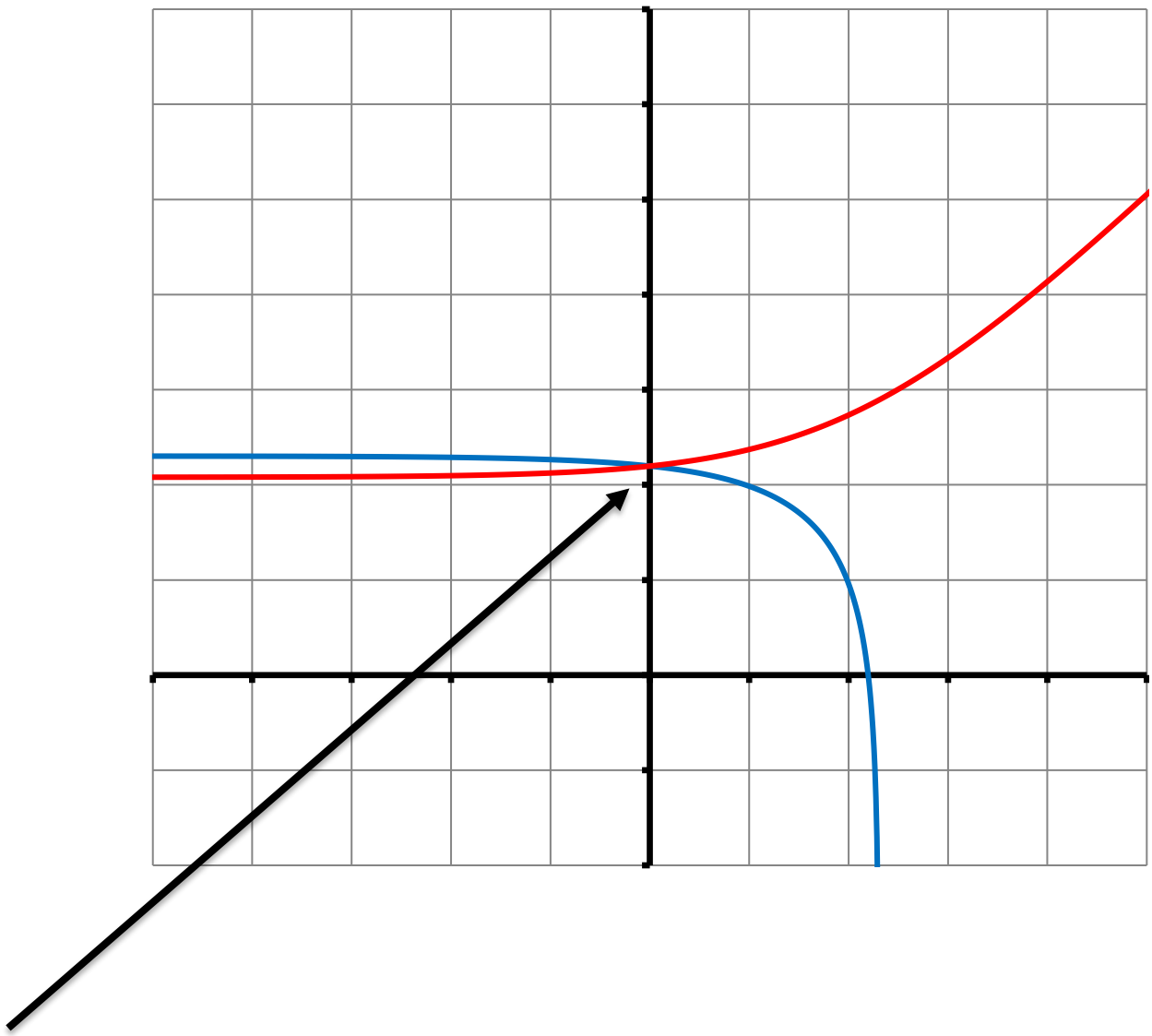
$$1 + e^y = 10$$

$$e^y = 9$$

$$y = \log_e(9)$$

$$y = \ln(9)$$

$$\mathbf{(0, \ln(9))}$$



$(0, \ln(9))$ is the only solution because that is the only point where the graphs intersect. Keep in mind that you can approximate $\ln(9)$ to verify that the point on the graph is correct ($\ln(9) \approx 2.197225\ldots$), but answers in LON-CAPA should be exact, so input $(0, \ln(9))$.

When working with systems of equations that involve exponential and/or logarithmic equations, it is often necessary to convert at least one equation from one form to the other. Be prepared to convert exponential equations to logarithmic form, like we did on Example 4, and be prepared to convert logarithmic equations to exponential form, like we'll do on Example 5. Feel free to review Lessons 33 and/or 34 for help on converting from one form to the other.

Example 5: Use the method of elimination to find all the real solutions for the following systems. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

$$\begin{cases} \log\left(\frac{1}{2}x - \frac{3}{5}y\right) = -1 \\ \log\left(x - \frac{1}{3}y\right) = 1 \end{cases}$$

To solve this system of equations using elimination, I will first convert from logarithmic form to exponential form.

$$\begin{cases} \log_{10}\left(\frac{1}{2}x - \frac{3}{5}y\right) = -1 \\ \log_{10}\left(x - \frac{1}{3}y\right) = 1 \end{cases}$$

$$\begin{cases} \frac{1}{2}x - \frac{3}{5}y = 10^{-1} \\ x - \frac{1}{3}y = 10^{1} \end{cases}$$

$$\begin{cases} \frac{1}{2}x - \frac{3}{5}y = \frac{1}{10} \\ x - \frac{1}{3}y = 10 \end{cases}$$

I now have a system of linear equations (all the variables have a degree of 1), but there are numerous fractions. To eliminate those fractions, I will multiply the first equation by 10 and the second equation by 3.

$$\begin{cases} 10\left(\frac{1}{2}x - \frac{3}{5}y\right) = \left(\frac{1}{10}\right)10 \\ 3\left(x - \frac{1}{3}y\right) = (10)3 \end{cases}$$

$$\begin{cases} 5x - 6y = 1 \\ 3x - y = 30 \end{cases}$$

Now that I have a system of equations with no fractions, I will solve it by elimination. To do so, I will multiply the second equation by -6, so that when the two equations are added together, the y terms will be eliminated.

$$\begin{cases} 5x - 6y = 1 \\ -6(3x - y) = -6(30) \end{cases}$$

$$\begin{cases} 5x - 6y = 1 \\ -18x + 6y = -180 \end{cases}$$

$$-13x = -179$$

$$x = \frac{179}{13}$$

Now I can back substitute to any of the prior equations in order to find the value of y. I will back substitute to the equation $x - \frac{1}{3}y = 10$.

$$\frac{179}{13} - \frac{1}{3}y = 10$$

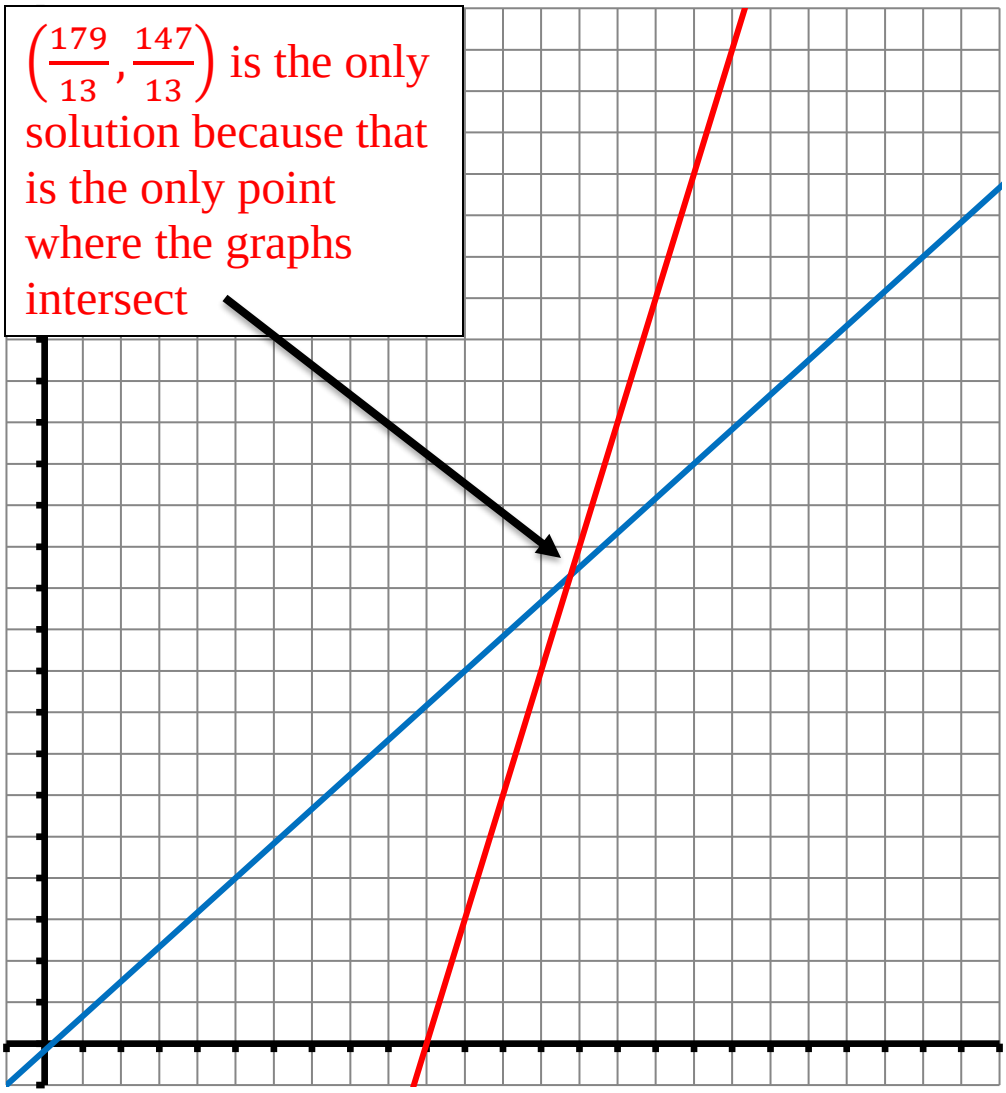
$$39\left(\frac{179}{13} - \frac{1}{3}y\right) = (10)39$$

$$537 - 13y = 390$$

$$-13y = -147$$

$$y = \frac{147}{13}$$

$$\left(\mathbf{\frac{179}{13}}, \mathbf{\frac{147}{13}}\right)$$



Since the original system contained logarithmic equations, we must check our answers to be sure that both arguments are positive:

$$\begin{cases} \log\left(\frac{1}{2}x - \frac{3}{5}y\right) = -1 \\ \log\left(x - \frac{1}{3}y\right) = 1 \end{cases}$$

Argument of Equation 1: $\frac{1}{2}\left(\frac{179}{13}\right) - \frac{3}{5}\left(\frac{147}{13}\right) = \frac{1}{10}$

Argument of Equation 2: $\frac{179}{13} - \frac{1}{3}\left(\frac{147}{13}\right) = 10$

This shows that both answers are valid.

<u>Example 6:</u> Solve each system and write your answers as an ordered pair. If the system has infinitely many solutions, write IMS; if there is no solution, write NS.

a. $\begin{cases} 9x + 2y = 0 \\ 3x - 5y = 17 \end{cases}$

b. $\begin{cases} \frac{1}{2}x + \frac{3}{5}y = 3 \\ \frac{5}{3}x + 2y = 10 \end{cases}$

c. $\begin{cases} 0.4x + 1.2y = 14 \\ 12x - 5y = 10 \end{cases}$

I will start by multiplying the first equation by 10 to eliminate the decimals.

$10(0.4x + 1.2y) = 10(14)$

$4x + 12y = 140$

Next, I'll multiply the new equation by -3 so that the x-coordinates have opposite coefficients.

$-12x - 36y = -420$

Now I'll add the new equation to the original second equation to eliminate the x-coordinates.

$$\begin{array}{r} -12x - 36y = -420 \\ + \underline{12x - 5y = 10} \\ 0 - 41y = -410 \end{array}$$

$-41y = -410$

$y = 10$

Now I can back substitute to any of the prior equations to solve for x.

$12x - 5y = 10$

$12x - 5(10) = 10$

$12x - 50 = 10$

$12x = 60$

$x = 5$

$\mathbf{(5, 10)}$

d. $\begin{cases} 2^{x+y} = 2 \\ 2^{x-y} = 1 \end{cases}$

I'll start by converting each equation to logarithmic form.

$\begin{cases} \log_2(2) = x + y \\ \log_2(1) = x - y \end{cases}$

$\begin{cases} 1 = x + y \\ 0 = x - y \end{cases}$

Now I'll simply add the new equations together in order to eliminate the y-coordinates .

$$\begin{array}{r} 1 = x + y \\ + \underline{0 = x - y} \\ 1 = 2x \end{array}$$

$\frac{1}{2} = x$

Now I can back substitute to any of the prior equations to solve for x.

$0 = x - y$

$0 = \frac{1}{2} - y$

$y = \frac{1}{2}$

$\mathbf{\left(\frac{1}{2}, \frac{1}{2}\right)}$

Answers to Examples:

6a. $\left(\frac{2}{3}, -3\right)$; *6b.* IMS ; *6c.* $(5,10)$; *6d.* $\left(\frac{1}{2}, \frac{1}{2}\right)$