There are three factoring formulas that will be used specifically for factoring binomials in this course. They are the difference of squares, the difference of cubes, and the sum of cubes. As the names indicate, we will be working with pairs of either perfect squares or perfect cubes that are either being added (sum) or subtracted (difference).

<u>Difference of Squares:</u>

- formula to factor two perfect squares that are being subtracted
  - $x^2 - y^2 = x \cdot x - y \cdot y$
    $= (x + y)(x - y)$
    $= (\text{factor one plus factor two})(\text{factor one minus factor two})$

  - $a^2 - 9$
    - notice in this binomial that both terms (a^2 and 9) are perfect squares (a^2 is $a \cdot a$, 9 is $3 \cdot 3$), so I will replace the x in the difference of squares factoring formula with a and I will replace the y in the formula with 3
    - since the perfect squares are being subtracted, this is a difference (subtraction) of squares
    - $a^2 - 9 = a \cdot a - 3 \cdot 3$

      $$= (\boldsymbol{a} + \mathbf{3})(\boldsymbol{a} - \mathbf{3})$$

  - $25a^8 - 16b^4$
    - once again this is a binomial in which both terms ($25a^8$ and $16b^4$) are perfect squares
    - $25a^8 - 16b^4 = 5a^4 \cdot 5a^4 - 4b^2 \cdot 4b^2$

      $$= (\mathbf{5}\boldsymbol{a}^{\mathbf{4}} + \mathbf{4}\boldsymbol{b}^{\mathbf{2}})(\mathbf{5}\boldsymbol{a}^{\mathbf{4}} - \mathbf{4}\boldsymbol{b}^{\mathbf{2}})$$

When factoring polynomials with real coefficients, there is no sum of squares formula

- $x^2 + 9$ is not factorable using real numbers
  - students will sometimes attempt to factor $x^2 + 9$ as $(x + 3)(x + 3)$; **THIS IS NOT CORRECT**
  - $(x + 3)(x + 3) = x^2 + 6x + 9$, **NOT** $x^2 + 9$
- $16x^2 + 25$ is another example of a perfect square plus a perfect square that is not factorable using real numbers

There is no sum of squares formula for factoring polynomials

Difference of Cubes:

- formula to factor two perfect cubes that are being subtracted
  - $x^3 - y^3 = x \cdot x \cdot x - y \cdot y \cdot y$
    $= (x - y)(x^2 + xy + y^2)$
  - $w^3 - 8$
    - both terms of this binomial (w^3 and 8) are perfect cubes (w^3 is $w \cdot w \cdot w$, 8 is $2 \cdot 2 \cdot 2$), and since they're being subtracted, this is a **difference of cubes**
    - using the difference of cubes formula, I will replace x with w and y with 2

Difference of Cubes Formula: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$$w^3 - 2^3 = \mathbf{(w - 2)(w^2 + 2w + 4)}$$

- $64a^6 - b^9$
  - once again this is a **difference of cubes**
  - $64a^6 - b^9 = 4a^2 \cdot 4a^2 \cdot 4a^2 - b^3 \cdot b^3 \cdot b^3$

Difference of Cubes Formula: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$$(4a^2)^3 - (b^3)^3 = (4a^2 - b^3)((4a^2)^2 + (4a^2)(b^3) + (b^3)^2)$$

$$= \mathbf{(4a^2 - b^3)(16a^4 + 4a^2b^3 + b^6)}$$

Sum of Cubes:

- formula to factor two perfect cubes that are being added
  - $x^3 + y^3 = x \cdot x \cdot x + y \cdot y \cdot y$
    $= (x + y)(x^2 - xy + y^2)$
  - $1 + 125z^3$
    - both terms of this binomial (1 and $125z^3$) are perfect cubes (1 is $1 \cdot 1 \cdot 1$, $125z^3$ is $5z \cdot 5z \cdot 5z$), and since they're being added, this is a **sum of cubes**
    - using the sum of cubes formula, I will replace x with 1 and y with $5z$

Sum of Cubes Formula: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$$1^3 + (5z)^3 = (1 + 5z)(1 - (1)(5z) + (5z)^2)$$

$$= \mathbf{(1 + 5z)(1 - 5z + 25z^2)}$$

  - $a^6 + b^6$
    - once again this is a **sum of cubes**
      - keep in mind that a^6 and b^6 are both perfect squares as well, but we do not have a sum of squares factoring formula
    - $a^6 + b^6 = a^2 \cdot a^2 \cdot a^2 + b^2 \cdot b^2 \cdot b^2$

Sum of Cubes Formula: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$$(a^2)^3 + (b^2)^3 = (a^2 + b^2)((a^2)^2 - (a^2)(b^2) + (b^2)^2)$$

$$= \mathbf{(a^2 + b^2)(a^4 - a^2b^2 + b^4)}$$

Both cube formulas will be provided in LON-CAPA on homework problems and exam problems, so don't worry about memorizing them. Instead, focus on being able to identify sums and differences of cubes and know how to use the formulas. For more examples of factoring binomials using the sum of cubes or difference of cubes formulas, take a look at Examples 1 and 2 on the following pages.

Example 1: Factor the polynomial $8a^6 - 27b^9$ completely.

$8a^6$ and $27b^9$ are both perfect cubes, and since those perfect cubes are being subtracted, this is a difference of cubes. To express this binomial as a difference of cubes more clearly, I will re-write $8a^6$ as $(2a^2)^3$ and re-write $(27b^9)$ as $(3b^3)^3$, that way I can actually see each term as a perfect cube.

$$8a^6 - 27b^9$$

$$(2a^2)^3 - (3b^3)^3$$

The difference of cubes formula is
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. The binomial we have is $8a^6 - 27b^9$, or $(2a^2)^3 - (3b^3)^3$. So using the difference of cubes formula I will replace all the x terms in that formula with $2a^2$ and all the y terms with $3b^3$:

$$x^3 \quad - \quad y^3 \quad = \quad (x \quad - \quad y)(\; x^2 \quad + \quad xy \quad + \quad y^2)$$

$$(2a^2)^3 - (3b^3)^3 = (2a^2 - 3b^3)((2a^2)^2 + (2a^2)(3b^3) + (3b^3)^2)$$

Now the only thing left to do is simplify terms like $(2a^2)^2$ and $(3b^3)^2$ by using the Product to a Power Rule, and to simplify the term $(2a^2)(3b^3)$ by multiplying those two monomials.

$$(2a^2)^3 - (3b^3)^3$$

$$(2a^2 - 3b^3)((2a^2)^2 + (2a^2)(3b^3) + (3b^3)^2)$$

$$\boldsymbol{(2a^2 - 3b^3)(4a^4 + 6a^2b^3 + 9b^6)}$$

Example 2: Factor the polynomial $z^6 + 64$ completely.

Since z^6 and 64 are both perfect cubes, and since those perfect cubes are being added, this is a sum of cubes. To express this binomial as a sum of cubes more clearly, I will re-write z^6 as $(z^2)^3$ and re-write (64) as $(4)^3$, that way I can actually see each term as a perfect cube.

$$(z^2)^3 + (4)^3$$

The difference of cubes formula is
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. So using the difference of cubes formula I will replace all the x terms in that formula with x^2 and all the y terms with 4:

$$x^3 \quad + \quad y^3 \quad = (x \quad + \quad y)(\,x^2 \quad - \quad xy \quad + \quad y^2)$$

$$(z^2)^3 + (4)^3 = (z^2 + 4)((z^2)^2 - (z^2)(4) + (4)^2)$$

$$\mathbf{(z^2 + 4)(z^4 - 4z^2 + 16)}$$

Example 3: Factor each polynomial completely.

a. $a^3 + 125$

b. $a^2 + b^2$

c. $64a^3 - b^6$

d. $25x^2 - 1$

Answers to Examples:

1. $(2x^2 - 3y^3)(4x^4 + 6x^2y^3 + 9y^6)$ *; 2.* $(x^2 + 4)(x^4 - 4x^2 + 16)$ *;*
3a. $(a + 5)(a^2 - 5a + 25)$ *; 3b.* $NOT\ FACTORABLE$ *;*
3c. $(4a - b^2)(16a^2 + 4ab^2 + b^4)$ *; 3d.* $(5x - 1)(5x + 1)$ *;*