AN INTRODUCTION TO LÉVY PROCESSES  
WITH APPLICATIONS IN FINANCE  
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Abstract. These notes aim at introducing Lévy processes in an informal and intuitive way. Several important results about Lévy processes, such as the Lévy-Khintchine formula, the Lévy-Itô decomposition and Girsanov’s transformations, are discussed. Applications of Lévy processes in financial modeling are presented and some popular models in finance are revisited from the point of view of Lévy processes.

1. Introduction – Motivation

Lévy processes play a fundamental role in Mathematical Finance, as well as in other fields of science, such as Physics (turbulence, laser cooling), Engineering (telecommunications, queues, dams) and the Actuarial science (insurance risk). A comprehensive overview of some applications of Lévy processes can be found in Prabhu (1998), in Barndorff-Nielsen, Mikosch, and Resnick (2001) and in Kyprianou (2006).

Lévy processes have become increasingly popular in Mathematical Finance because they can describe the observed reality of financial markets in a more accurate way than models based on Brownian motion. In the “real” world, we observe that asset price processes have jumps or spikes, see Figure 1.1, and risk-managers have to take them into account. Moreover, the empirical distribution of asset returns exhibits fat tails and skewness, behavior that deviates from normality, see Figure 1.2. Hence, models that accurately fit return distributions are essential for the estimation of profit and loss (P&L) distributions. In the “risk-neutral” world, we observe that implied volatilities are constant neither across strike nor across maturities as stipulated by the Black and Scholes (1973) (actually, Samuelson 1965) model, see Figure 1.3. Therefore, traders need models that can capture the behavior of the implied volatility smiles more accurately, in order to handle the risk of trades. Lévy processes provide us with the appropriate framework to adequately describe all these observations, both in the “real” and in the “risk-neutral” world.

Paul Lévy. Processes with independent and stationary increments are named Lévy processes after the French mathematician Paul Lévy (1886-1971), who made the connection with infinitely divisible laws, characterized their distributions (Lévy-Khintchine formula) and described their structure.
Figure 1.1. USD/JPY foreign exchange rate, October 1997-October 2004.

Figure 1.2. Empirical distribution of daily log-returns for the GBP/USD exchange rate and fitted Normal distribution.

(Levy-Itô decomposition). Paul Lévy is one of the founding fathers of the theory of stochastic processes and made major contributions to the field of probability theory. Among others, Paul Lévy contributed to the study of Gaussian variables and processes, the law of large numbers, the central limit theorem, stable laws, infinitely divisible laws and pioneered the study of processes with independent and stationary increments.

More information about Paul Lévy and his scientific work, can be found at the websites

http://www.cmap.polytechnique.fr/~rama/levy.html

and

http://www.annales.org/archives/x/paullevy.html

(in French).
2. Definition

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a filtered probability space, where \(\mathcal{F} = \mathcal{F}_T\) and the filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}\) satisfies the usual conditions (cf. Jacod and Shiryaev 2003, I.1.3). Note that, by abuse of the standard notation, whenever we write \(t \geq 0\) that means \(t \in [0,T]\).

**Definition 2.1.** A càdlàg, adapted, real valued stochastic process \(L = (L_t)_{t \geq 0}\) with \(L_0 = 0\) a.s. is called a Lévy process if the following conditions are satisfied:

\((L1)\): \(L_t\) has independent increments, i.e. \(L_t - L_s\) is independent of \(\mathcal{F}_s\) for any \(0 \leq s < t \leq T\).

\((L2)\): \(L_t\) has stationary increments, i.e. for any \(s, t \geq 0\) the distribution of \(L_{t+s} - L_t\) does not depend on \(t\).

\((L3)\): \(L_t\) is stochastically continuous, i.e for every \(t \geq 0\) and \(\epsilon > 0\):

\[
\lim_{s \to t} P(|L_t - L_s| > \epsilon) = 0.
\]

The simplest Lévy process is the linear drift, a deterministic process. Brownian motion is the only (non-deterministic) Lévy process with continuous sample paths. Other examples of Lévy processes are the Poisson and compound Poisson processes. Notice that the sum of a Brownian motion and a compound Poisson process is again a Lévy process; it is often called a “jump-diffusion” process; we shall call it a “Lévy jump-diffusion” process, because there exist jump-diffusion processes which are not Lévy processes.

3. First example: a Lévy jump-diffusion

Assume that the process \(L = (L_t)_{t \geq 0}\) is a Lévy jump-diffusion, i.e. a Brownian motion plus a compensated compound Poisson process. It can be
Figure 2.4. Examples of Lévy processes: linear drift (left) and Brownian motion

Figure 2.5. Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion

described by

\[
L_t = bt + \sigma W_t + \left( \sum_{k=1}^{N_t} J_k - t\lambda \kappa \right)
\]

where \( b \in \mathbb{R}, \sigma \in \mathbb{R}_+, \) \( W = (W_t)_{t \geq 0} \) is a standard Brownian motion, \( N = (N_t)_{t \geq 0} \) is a Poisson process with parameter \( \lambda \) (i.e. \( \mathbb{E}[N_t] = \lambda t \)) and \( J = (J_k)_{k \geq 1} \) is an i.i.d. sequence of random variables with probability distribution \( F \) and \( \mathbb{E}[J] = \kappa < \infty; \) \( F \) describes the distribution of jump size. All sources of randomness are mutually independent.

It is well known that Brownian motion is a martingale; moreover, the compensated compound Poisson process is a martingale. Therefore, \( L = (L_t)_{t \geq 0} \) is a martingale if and only if \( b = 0. \)

The characteristic function of \( L_t \) is

\[
\mathbb{E}[e^{iuL_t}] = \mathbb{E}\left[\exp\left(\imath u (bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda \kappa)\right)\right]
\]

\[
= \exp\left(\imath ubt \right) \mathbb{E}\left[\exp\left(\imath u \sigma W_t \right) \exp\left(\imath u \left( \sum_{k=1}^{N_t} J_k - t\lambda \kappa \right)\right)\right];
\]
since all the sources of randomness are independent, we get

$$= \exp \left[ i \omega t \right] \mathbb{E} \left[ \exp \left( i \omega \sigma W_t \right) \right] \mathbb{E} \left[ \exp \left( i \omega \sum_{k=1}^{N_t} J_k - i \omega t \lambda \right) \right];$$

taking into account that

$$\mathbb{E} \left[ e^{i \omega W_t} \right] = e^{-\frac{1}{2} \sigma^2 \omega^2 t}, \quad W_t \sim \text{Normal}(0, \sigma^2 t)$$

$$\mathbb{E} \left[ e^{i \omega \sum_{k=1}^{N_t} J_k} \right] = e^{\lambda \mathbb{E} \left[ e^{i \omega J - 1} \right]}, \quad N_t \sim \text{Poisson}(\lambda t)$$

(cf. also Appendix B) we get

$$= \exp \left[ i \omega t \right] \exp \left[ -\frac{1}{2} \omega^2 \sigma^2 t \right] \exp \left[ \lambda \mathbb{E} \left[ e^{i \omega J - 1} \right] \right] \exp \left[ \lambda \mathbb{E} \left[ e^{i \omega J - 1} - i \omega \mathbb{E} J \right] \right];$$

and because the distribution of \( J \) is \( F \) we have

$$= \exp \left[ i \omega t \right] \exp \left[ -\frac{1}{2} \omega^2 \sigma^2 t \right] \exp \left[ \lambda t \int_{\mathbb{R}} \left( e^{i \omega x} - 1 - i \omega x \right) F(dx) \right].$$

Now, since \( t \) is a common factor, we re-write the above equation as

$$E \left[ e^{i \omega L_t} \right] = \exp \left[ \left( i \omega b - \frac{\omega^2 \sigma^2}{2} \right) t + \lambda \int_{\mathbb{R}} \left( e^{i \omega x} - 1 - i \omega x \right) \lambda F(dx) \right].$$

Since the characteristic function of a random variable determines its distribution, we have a “characterization” of the distribution of the random variables underlying the Lévy jump-diffusion. We will soon see that this distribution belongs to the class of infinitely divisible distributions. We will also see that equation (3.2) is a special case of the celebrated Lévy-Khintchine formula.

4. INFINITELY DIVISIBLE DISTRIBUTIONS AND THE LÉVY-KHINTCHINE FORMULA

There is strong interplay between Lévy processes and infinitely divisible distributions. We first define infinitely divisible distributions and give some examples and then describe their relationship to Lévy processes.

Let \( X \) be a real valued random variable, denote its characteristic function by \( \varphi_X \) and its law by \( P_X \), hence \( \varphi_X(u) = \int_{\mathbb{R}} e^{iux} P_X(dx) \).

**Definition 4.1.** The law \( P_X \) of a random variable \( X \) is **infinitely divisible**, if for all \( n \in \mathbb{N} \) there exist i.i.d. random variables \( X_1^{(1/n)}, \ldots, X_n^{(1/n)} \) such that

$$X \stackrel{d}{=} X_1^{(1/n)} + \ldots + X_n^{(1/n)}.$$  

Equivalently, the law \( P_X \) of a random variable \( X \) is **infinitely divisible** if for all \( n \in \mathbb{N} \) there exists another law \( P_X^{(1/n)} \) of a random variable \( X^{(1/n)} \) such
that

\[ P_X = P_{X^{(1/n)}} \ast \ldots \ast P_{X^{(1/n)}} \]  

(4.2)

Alternatively, we can characterize an infinitely divisible random variable using its characteristic function.

**Definition 4.2.** The law of a random variable \( X \) is **infinitely divisible**, if for all \( n \in \mathbb{N} \), there exists a random variable \( X^{(1/n)} \), such that

\[ \phi_X(u) = \left( \phi_{X^{(1/n)}}(u) \right)^n. \]  

(4.3)

**Example 4.3 (Normal distribution).** Using the second definition, we can easily see that the Normal distribution is infinitely divisible. Let \( X \sim \text{Normal}(\mu, \sigma^2) \), then we have

\[
\begin{align*}
\phi_X(u) &= \exp \left[ iu \mu - \frac{1}{2} u^2 \sigma^2 \right] \\
&= \exp \left[ iun \frac{\mu}{n} - \frac{1}{2} u^2 n \frac{\sigma^2}{n} \right] \\
&= \exp \left[ n \left( iun \frac{\mu}{n} - \frac{1}{2} u^2 \sigma^2 \right) \right] \\
&= \left( \exp \left[ iu \frac{\mu}{n} - \frac{1}{2} u^2 \frac{\sigma^2}{n} \right] \right)^n \\
&= \left( \phi_{X^{(1/n)}}(u) \right)^n 
\end{align*}
\]

where \( X^{(1/n)} \sim \text{Normal}(\frac{\mu}{n}, \frac{\sigma^2}{n}) \).

**Example 4.4 (Poisson distribution).** Following the same procedure, we can easily deduce that the Poisson distribution is infinitely divisible. Let \( X \sim \text{Poisson}(\lambda) \), then we have

\[
\begin{align*}
\phi_X(u) &= \exp \left[ \lambda (e^{iu} - 1) \right] \\
&= \exp \left[ n \frac{\lambda}{n} (e^{iu} - 1) \right] \\
&= \left( \exp \left[ \frac{\lambda}{n} (e^{iu} - 1) \right] \right)^n \\
&= \left( \phi_{X^{(1/n)}}(u) \right)^n 
\end{align*}
\]

where \( X^{(1/n)} \sim \text{Poisson}(\frac{\lambda}{n}) \).

**Remark 4.5.** Other examples of infinitely divisible distributions are the compound Poisson distribution, the exponential, the \( \Gamma \)-distribution, the geometric, the negative binomial, the Cauchy distribution and the strictly stable distribution. Counter-examples are the uniform and binomial distributions.

The next theorem provides a complete characterization of random variables with infinitely divisible distributions via their characteristic functions; this is the celebrated *Lévy-Khintchine formula*. 

Theorem 4.6. The law $P_X$ of a random variable $X$ is infinitely divisible if and only if there exists a triplet $(b, c, \nu)$, with $b \in \mathbb{R}$, $c \in \mathbb{R}_+$ and a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty$, such that

$$
\mathbb{E}[e^{iuX}] = \exp \left[ i bu - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbb{1}_{\{|x|<1\}}) \nu(dx) \right].
$$

(4.4)


The triplet $(b, c, \nu)$ is called the Lévy or characteristic triplet and the exponent in (4.4)

$$
\psi(u) = iub - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbb{1}_{\{|x|<1\}}) \nu(dx)
$$

(4.5)

is called the Lévy or characteristic exponent. Moreover, $b \in \mathbb{R}$ is called the drift term, $c \in \mathbb{R}_+$ the Gaussian or diffusion coefficient and $\nu$ the Lévy measure.

Now, consider a Lévy process $L = (L_t)_{t \geq 0}$; using the fact that, for any $n \in \mathbb{N}$ and any $t > 0$,

$$
L_t = L_{t/n} + (L_{2t/n} - L_{t/n}) + \ldots + (L_t - L_{(n-1)t/n})
$$

(4.6)

together with the stationarity and independence of the increments, we conclude that the random variable $L_t$ is infinitely divisible.

Moreover, for all $u \in \mathbb{R}$ and all $t \geq 0$, define

$$
\psi_t(u) = \log \mathbb{E}[e^{iuL_t}];
$$

(4.7)

making use of (4.6) and the stationarity and independence of the increments, we have for any $m \in \mathbb{N}$

$$
m\psi_1(u) = \psi_m(u)
$$

(4.8)

and hence, for any rational $t > 0$

$$
t\psi_1(u) = \psi_t(u).
$$

(4.9)

For an irrational $t$, we can choose a sequence of rationals that decreases to $t$ and use right continuity of $L$ to prove that (4.9) holds for all $t \geq 0$.

Therefore, we have that for every Lévy process, the following property holds

$$
\mathbb{E}[e^{iuL_t}] = e^{t\psi(u)}
$$

(4.10)

where $\psi(u) := \psi_1(u)$ is the characteristic exponent of $L_1 := X$ (say), a random variable with an infinitely divisible distribution.

We have seen so far, that every Lévy process can be associated with the law of an infinitely divisible distribution. The opposite, i.e. that given any random variable $X$, whose law is infinitely divisible, we can construct a Lévy process $L = (L_t)_{t \geq 0}$ such that $L_1 := X$, is also true. This will be the subject of the next section.
The Lévy-Itô decomposition

Theorem 5.1. Consider a triplet \((b, c, \nu)\) where \(b \in \mathbb{R}\), \(c \in \mathbb{R}^+\) and \(\nu\) is a measure satisfying \(\nu(\{0\}) = 0\) and \(\int_\mathbb{R}(1 \wedge |x|^2)\nu(dx) < \infty\). Then, there exists a probability space \((\Omega, \mathcal{F}, P)\) on which four independent Lévy processes exist, \(L^{(1)}\), \(L^{(2)}\), \(L^{(3)}\) and \(L^{(4)}\), where \(L^{(1)}\) is a constant drift, \(L^{(2)}\) is a Brownian motion, \(L^{(3)}\) is a compound Poisson process and \(L^{(4)}\) is a square integrable (pure jump) martingale with an a.s. countable number of jumps on each finite time interval of magnitude less that 1. Taking \(L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}\), we have that there exists a probability space on which a Lévy process \(L = (L_t)_{t \geq 0}\) with characteristic exponent

\[
\psi(u) = iub - \frac{u^2c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{|x|<1})\nu(dx) \quad (5.1)
\]

for all \(u \in \mathbb{R}\), is defined.


The Lévy-Itô decomposition is a hard mathematical result to prove; here, we go through some steps of the proof because it reveals much about the structure of the paths of a Lévy process. We split the Lévy exponent (5.1) into four parts

\[
\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \psi^{(4)}
\]

where

\[
\psi^{(1)}(u) = iub, \quad \psi^{(2)}(u) = \frac{u^2c}{2},
\]

\[
\psi^{(3)}(u) = \int_{|x| \geq 1} (e^{iux} - 1)\nu(dx),
\]

\[
\psi^{(4)}(u) = \int_{|x| < 1} (e^{iux} - 1 - iux)\nu(dx).
\]

The first part corresponds to a deterministic linear process (drift) with parameter \(b\), the second one to a Brownian motion with coefficient \(c\) and the third part to the characteristic function of a compound Poisson process with arrival rate \(\lambda := \nu(\mathbb{R} \setminus (-1, 1))\) and jump magnitude \(F(dx) := \nu(dx)\mathbb{1}_{|x| \geq 1}\).

The last part is the most difficult to handle; let \(\Delta L^{(4)}\) denote the jumps of the Lévy process \(L^{(4)}\) and \(\mu^{L^{(4)}}\) denote the random measure counting the jumps of \(L^{(4)}\). Next, one constructs a compensated compound Poisson process

\[
L^{(4,c)}_t = \sum_{0 \leq s \leq t} \Delta L^{(4)}_s \mathbb{1}_{1>\epsilon} - t\left( \int_{1>\epsilon} x\nu(dx) \right) = \int_0^t \int_{1>\epsilon} x\mu^{L^{(4)}}(dx, ds) - t\left( \int_{1>\epsilon} x\nu(dx) \right)
\]
and shows that the jumps of $L^{(4)}$ form a Poisson point process; using results for Poisson point processes, we get that the characteristic function of $L^{(4,\epsilon)}$ is

$$\psi^{(4,\epsilon)}(u) = \int_{\epsilon < |x| < 1} (e^{iux} - 1 - iux)\nu(dx).$$

Finally, there exists a Lévy process $L^{(4)}$ which is a square integrable martingale and $L^{(4,\epsilon)} \to L^{(4)}$ uniformly on $[0, T]$ as $\epsilon \to 0^+$ with Lévy exponent $\psi^{(4)}$.

Therefore, we can decompose any Lévy process into four independent Lévy processes $L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$, i.e.

$$L_t = bt + \sqrt{c}W_t + \int_0^t \int_{|x| \geq 1} x\mu(ds,dx)$$

$$+ \left( \int_0^t \int_{|x| < 1} x\mu(ds,dx) - t \int_{|x| < 1} x\nu(dx) \right)$$

where $L^{(1)}$ is a constant drift, $L^{(2)}$ a Brownian motion, $L^{(3)}$ a compound Poisson process and $L^{(4)}$ a pure jump martingale. This result is the celebrated Lévy-Itô decomposition of a Lévy process.

6. The Lévy Measure and Path Properties

The Lévy measure $\nu$ is a measure on $\mathbb{R}$ that satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (1 \wedge |x|^2)\nu(dx) < \infty.$$  

That means, that a Lévy measure has no mass at the origin, but singularities (i.e. infinitely many jumps) can occur around the origin (i.e. small jumps). Intuitively speaking, the Lévy measure describes the expected number of jumps of a certain height in a time interval of length 1.

For example, the Lévy measure of the Lévy jump-diffusion is $\nu(dx) = \lambda \cdot F(dx)$; from that we can deduce that the expected number of jumps, in a time interval of length 1, is $\lambda$ and the jump size is distributed according to $F$.

More generally, if $\nu$ is a finite measure, i.e. $\nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(dx) = \lambda < \infty$, then $F(dx) := \frac{\nu(dx)}{\lambda}$, which is a probability measure. Thus, $\lambda$ is the expected number of jumps and $F(dx)$ the distribution of the jump size $x$. If $\nu(\mathbb{R}) = \infty$, then an infinite number of (small) jumps is expected.

The Lévy measure is responsible for the richness of the class of Lévy processes and carries useful information about the structure of the process. Path properties of a Lévy process can be read from the Lévy measure; for example, Figures 6.6 and 6.7 reveal that the compound Poisson process has a finite number of jumps on every time interval, while the NIG and $\alpha$-stable processes have an infinite one; we then speak of an infinite activity Lévy process.
Figure 6.6. The distribution function of the Lévy measure of the standard Poisson process (left) and the density of the Lévy measure of a compound Poisson process with double-exponentially distributed jumps.

Figure 6.7. The density of the Lévy measure of an NIG (left) and an α-stable process.

Proposition 6.1. Let $L$ be a Lévy process with triplet $(b, c, ν)$.

(1) If $ν(\mathbb{R}) < \infty$ then almost all paths of $L$ have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.

(2) If $ν(\mathbb{R}) = \infty$ then almost all paths of $L$ have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.


Whether a Lévy process has finite variation or not also depends on the Lévy measure (and on the presence or absence of a Brownian part).

Proposition 6.2. Let $L$ be a Lévy process with triplet $(b, c, ν)$.

(1) If $c = 0$ and $\int_{|x| \leq 1} |x| ν(dx) < \infty$ then almost all paths of $L$ have finite variation.

(2) If $c \neq 0$ or $\int_{|x| \leq 1} |x| ν(dx) = \infty$ then almost all paths of $L$ have infinite variation.


The different functions a Lévy measure has to integrate in order to have finite activity or variation, are graphically exhibited in Figure 6.8. The
compound Poisson process has finite measure, *hence* it has finite variation as well; on the contrary, the NIG has an infinite measure *and* has infinite variation.

Moreover, the Lévy measure carries information about the finiteness of the *moments* of a Lévy process; this is extremely useful information in mathematical finance, for the existence of a *martingale measure*. The finiteness of the moments of a Lévy process is related to the finiteness of an integral over the Lévy measure (more precisely, the restriction of the Lévy measure to jumps larger than 1 in absolute value, i.e. big jumps).

**Proposition 6.3.** Let $L$ be a Lévy process with triplet $(b, c, \nu)$. Then

1. $L_t$ has finite $p$-th moment for $p \in \mathbb{R}_+$ $(\mathbb{E}|L_t|^p < \infty)$ if and only if $\int_{|x| \geq 1} |x|^p \nu(dx) < \infty$.

2. $L_t$ has finite $p$-th exponential moment for $p \in \mathbb{R}$ $(\mathbb{E}[e^{pL_t}] < \infty)$ if and only if $\int_{|x| \geq 1} e^{px} \nu(dx) < \infty$.

**Proof.** The proof of these results can be found in Theorem 25.3 in Sato (1999). Actually, the conclusion of this theorem holds for the general class of *submultiplicative* functions (cf. Definition 25.1 in Sato 1999), which contains $\exp(px)$ and $|x|^p \vee 1$ as special cases.

In order to gain some understanding of this result and because it blends beautifully with the Lévy-Itô decomposition, we will give a rough proof of the sufficiency for the second part (inspired by Kyprianou 2006).

Recall from the Lévy-Itô decomposition, that the characteristic exponent of a Lévy process was split into four independent parts, the third of which is a compound Poisson process with arrival rate $\lambda := \nu(\mathbb{R} \setminus (-1, 1))$ and jump magnitude $F(dx) := \frac{\nu(dx)}{\nu(\mathbb{R} \setminus (-1, 1))} \mathbb{1}_{\{|x| \geq 1\}}$. Finiteness of $\mathbb{E}[e^{pL_t}]$ implies
finiteness of $\mathbb{E}[e^{pL^{(3)}_t}]$, where

$$
\mathbb{E}[e^{pL^{(3)}_t}] = e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t)^k}{k!} \left( \int_{\mathbb{R}} e^{px} F(dx) \right)^k
$$

$$
= e^{-\lambda t} \sum_{k \geq 0} \frac{t^k}{k!} \left( \int_{\mathbb{R}} e^{px} \mathbb{1}_{|x| \geq 1} \nu(dx) \right)^k.
$$

Since all the summands are finite, the one corresponding to $k = 1$ must also be finite, therefore

$$
e^{-\lambda t} \int_{\mathbb{R}} e^{px} \mathbb{1}_{|x| \geq 1} \nu(dx) < \infty \implies \int_{|x| \geq 1} e^{px} \nu(dx) < \infty.
$$

![Figure 6.9](image)

Figure 6.9. A Lévy process has first moment if the Lévy measure integrates $|x|$ for $|x| \geq 1$ (blue line) and second moment if it integrates $x^2$ for $|x| \geq 1$ (orange line).

The graphical representation of the functions the Lévy measure must integrate so that a Lévy process has finite moments is given in Figure 6.9. The NIG process possesses moments of all order, while the $\alpha$-stable does not; one can already see from Figure 6.7 that the tails of the Lévy measure of the $\alpha$-stable are heavier than that of the NIG.

7. Some cases of particular interest

We already know that a Brownian motion, a (compound) Poisson process and a Lévy jump-diffusion are Lévy processes, their Lévy-Itô decomposition and their characteristic functions. Here, we present some other special cases of Lévy processes that are of special interest.

7.1. Subordinator. A subordinator is an a.s. increasing (in $t$) Lévy process. Equivalently, for $L$ to be a subordinator, the triplet must satisfy $\nu(-\infty, 0) = 0$, $c = 0$, $\int_{(0,1)} x \nu(dx) < \infty$ and $\gamma = b + \int_{(0,1)} x \nu(dx) > 0$. 
The Lévy-Itô decomposition of a subordinator is

\begin{equation}
L_t = \gamma t + \int_0^t \int_{(0,\infty)} x \mu^L(ds, dx)
\end{equation}

and the Lévy-Khintchine formula takes the form

\begin{equation}
\mathbb{E}[e^{iuL_t}] = \exp \left[ t(iu\gamma + \int_{(0,\infty)} (e^{iux} - 1) \nu(dx)) \right].
\end{equation}

Two examples of subordinators are the Poisson and the inverse Gaussian process, cf. Figures 7.10 and A.13.

7.2. Jumps of finite variation. A Lévy process has jumps of finite variation if and only if \( \int_{|x|\leq 1} |x| \nu(dx) < \infty \). In this case, the Lévy-Itô decomposition of \( L \) resumes the form

\begin{equation}
L_t = \gamma t + \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}} x \mu^L(ds, dx)
\end{equation}

and the Lévy-Khintchine formula takes the form

\begin{equation}
\mathbb{E}[e^{iuL_t}] = \exp \left[ t(iu\gamma - \frac{u^2c}{2} + \int_{\mathbb{R}} (e^{iux} - 1) \nu(dx)) \right],
\end{equation}

where \( \gamma \) is defined as above.

Moreover, if \( \nu([-1, 1]) < \infty \), which means that \( \nu(\mathbb{R}) < \infty \) then \( L \) is a compound Poisson process.

7.3. Spectrally one-sided. A Lévy processes is called spectrally negative if \( \nu(0, \infty) = 0 \). Similarly, a Lévy processes is called spectrally positive if \( -L \) is spectrally negative.

7.4. Finite first moment. As we have seen already, a Lévy process has a finite first moment if and only if \( \int_{|x|\geq 1} |x| \nu(dx) < \infty \). Therefore, the Lévy-Itô decomposition of \( L \) resumes the form

\begin{equation}
L_t = b't + \sqrt{c} W_t + \left( \int_0^t \int_{\mathbb{R}} x \mu^L(ds, dx) - t \int_{\mathbb{R}} x \nu(dx) \right)
\end{equation}

and the Lévy-Khintchine formula takes the form

\begin{equation}
\mathbb{E}[e^{iuL_t}] = \exp \left[ t(ibu' - \frac{u^2c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux) \nu(dx)) \right],
\end{equation}

where \( b' = b + \int_{|x|\geq 1} x \nu(dx) \).

Remark 7.1 (Assumption (M)). For the remaining parts we will work only with Lévy process that have a finite first moment. We will refer to them as Lévy processes that satisfy assumption (M). For the sake of simplicity, we suppress the notation \( b' \) and write \( b \) instead.
8. Elements from semimartingale theory

A semimartingale is a stochastic process $X$ which admits the decomposition

$$X = X_0 + M + A$$

where $X_0$ is finite and $\mathcal{F}_0$-measurable, $M$ is a local martingale with $M_0 = 0$ and $A$ is a finite variation process with $A_0 = 0$. $X$ is a special semimartingale if $A$ is predictable.

Any special semimartingale $X$ has the following, so-called, canonical decomposition

$$X_t = X_0 + B_t + X^c_t + \int_0^t \int _0^\infty x(\mu^X_t - \nu^X_t)(ds, dx)$$

where $X^c_t$ is the continuous martingale part of $X$ and $\int_0^t \int _0^\infty x(\mu^X_t - \nu^X_t)(ds, dx)$ is the purely discontinuous martingale part of $X$. $\mu^X_t$ is called the random measure of jumps of $X$; it counts the number of jumps of specific size that occur in a time interval of specific length. $\nu^X_t$ is called the compensator of $\mu^X$ (for more details, see chapter II in Jacod and Shiryaev 2003).

Returning to the Lévy-Itô decomposition (5.2), we can easily see that a Lévy process with triplet $(b, c, \nu)$ which satisfies assumption $(\mathcal{M})$, has the following canonical decomposition

$$L_t = bt + \sqrt{c}W_t + \int_0^t \int _0^\infty x(\mu^L_t - \nu^L_t)(ds, dx),$$

where

$$\int_0^t \int _0^\infty x\mu^L_t(ds, dx) = \sum_{0 \leq s \leq t} \Delta L_s$$

and

$$\mathbb{E}\left[ \int_0^t \int _0^\infty x\mu^L_t(ds, dx) \right] = \int_0^t \int _0^\infty x\nu^L_t(ds, dx) = t \int _0^\infty x\nu(dx);$$

Figure 7.10. Simulated path of a normal inverse Gaussian (left) and an inverse Gaussian process.
one also writes $\nu^L(ds, dx) = \nu(dx) ds$.

We denote the continuous martingale part of $L$ by $L^c$ and the purely discontinuous martingale part of $L$ by $L^d$, i.e.

$$L^c_t = \sqrt{c} W_t \quad \text{and} \quad L^d_t = \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^L)(ds, dx).$$

(8.1)

Remark 8.1. Every Lévy process is also a semimartingale. Every Lévy process with finite first moment (i.e. satisfies assumption (M)) is also a special semimartingale; conversely, every Lévy process that is a special semimartingale, has a finite first moment.

9. Girsanov’s theorem

We will describe a special case of Girsanov’s theorem for semimartingales, where a Lévy process remains a process with independent increments (PII) under the new measure.

Let $P$ and $\tilde{P}$ be probability measures defined on the filtered space $(\Omega, F, F)$. Two probability measures $P$ and $\tilde{P}$ are equivalent measures, if $P(A) = 0 \iff \tilde{P}(A) = 0$ and one writes $P \sim \tilde{P}$.

Whenever $P \sim \tilde{P}$, there exists a unique positive $P$-martingale $\eta = (\eta_t)_{t \geq 0}$ such that $\mathbb{E}[\frac{d\tilde{P}}{dP}|F_t] = \eta_t$, $\forall t \geq 0$ (the Radon-Nikodym derivative); $\eta$ is called the density process of $\tilde{P}$ with respect to $P$.

Conversely, given a measure $P$ and a positive $P$-martingale $\eta = (\eta_t)_{t \geq 0}$, one can define a measure $\tilde{P}$ on $(\Omega, F)$, where $P \sim \tilde{P}$, via the Radon-Nikodym derivative $\mathbb{E}[\frac{d\tilde{P}}{dP}|F_T] = \eta_T$.

Theorem 9.1. Let $L$ be a Lévy process with triplet $(b, c, \nu)$ under $P$, that satisfies (M). It has the canonical decomposition

$$L_t = bt + \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^L)(ds, dx).$$

(9.1)

(A1): Assume that $P \sim \tilde{P}$. Then, there exist a deterministic process $\beta$ and a measurable non-negative deterministic process $Y$, satisfying

$$\int_0^t \int_{\mathbb{R}} |x(Y(s, x) - 1)| \nu(dx) ds < \infty,$$

(9.2)

and

$$\int_0^t (c \cdot \beta_s^2) ds < \infty$$
such that the density process $\eta = (\eta_t)_{t \geq 0}$ has the form

$$
\eta_t = \mathbb{E} \left[ \frac{d\tilde{P}}{dP} \bigg| \mathcal{F}_t \right] = \exp \left[ t \int_0^t \beta_s \sqrt{c} \, dW_s - \frac{1}{2} \int_0^t \beta_s^2 \, ds 
+ \int_0^t \int_\mathbb{R} (Y(s, x) - 1)(\mu^L - \nu^L)(ds, dx) 
- \int_0^t \int_\mathbb{R} (Y(s, x) - 1 - \ln(Y(s, x))) \mu^L(ds, dx) \right].
$$

(A2): Conversely, if $\eta$ is a positive martingale of the form (9.3), then it defines a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$, such that $P \sim \tilde{P}$.

(A3): In both cases, we have that $\tilde{W}_t = W_t - \int_0^t \sqrt{c} \beta_s \, ds$ is a $\tilde{P}$-Brownian motion, $\nu^L(ds, dx) = Y(s, x) \nu^L(ds, dx)$ is the $P$-compensator of $\mu^L$ and $L$ has the following canonical decomposition under $\tilde{P}$

$$
L_t = \tilde{b} t + \sqrt{c} \tilde{W}_t + \int_0^t \int_\mathbb{R} x(\mu^L - \nu^L)(ds, dx),
$$

where

$$
\tilde{b} t = b t + \int_0^t c \beta_s \, ds + \int_0^t \int_\mathbb{R} x(Y(s, x) - 1) \nu^L(ds, dx).
$$

Proof. Theorems III.3.24, III.5.19 and III.5.35 in Jacod and Shiryaev (2003) yield the result. \hfill \Box

Remark 9.2. Notice that from condition (9.2) follows that $L$ has finite first moment under $\tilde{P}$ as well, i.e.

$$
\mathbb{E}[L_t] < \infty, \quad \text{for all } t \geq 0.
$$

Remark 9.3. The process $L$ is not necessarily a Lévy process under the new measure $\tilde{P}$; it depends on the tuple $(\beta, Y)$. We have the following cases

(G1): if $(\beta, Y)$ are deterministic and independent of time, then $L$ remains a Lévy process under $\tilde{P}$; its triplet is $(\tilde{b}, c, Y \cdot \nu)$.

(G2): if $(\beta, Y)$ are deterministic but depend on time, then $L$ becomes a process with independent (but not stationary) increments under $\tilde{P}$, often called an additive process.

(G3): if $(\beta, Y)$ are neither deterministic nor independent of time, then $L$ is a semimartingale under $\tilde{P}$.

Remark 9.4. Notice that $c$, the diffusion coefficient, and $\mu^L$, the random measure of jumps of $L$, did not change under the change of measure from $P$ to $\tilde{P}$. That happens because $c$ and $\mu^L$ are path properties of the process and do not change under an equivalent change of measure. Intuitively speaking, the paths do not change, the probability of certain paths occurring changes.
Example 9.5. Assume that $L$ is a Lévy process with canonical decomposition (9.1) under $P$. Assume that $P \sim \tilde{P}$ and the density process is
\begin{equation}
\eta_t = \exp \left[ \beta \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}} \alpha x (\mu^L - \nu^L)(ds, dx) \right]
- \left( c \beta^2 + \int_{\mathbb{R}} (e^{\alpha x} - 1 - \alpha x) \nu(dx) \right) t,
\end{equation}
where $\beta \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ are constants.

Then, comparing (9.6) with (9.3), we have that the tuple of functions that characterize the change of measure is $(\beta, Y) = (\beta, f)$, where $f(x) = e^{\alpha x}$. Because $(\beta, Y)$ are deterministic and independent of time, $L$ remains a Lévy process under $\tilde{P}$, its Lévy triplet is $(\tilde{b}, c, e^{\alpha x} \nu)$ and its canonical decomposition is given by equations (9.4) and (9.5).

Remark 9.6. Actually, the change of measure of example 9.5 corresponds to the so-called Esscher transformation or exponential tilting. In chapter 3 of Kyprianou (2006), one can find a significantly easier proof of Girsanov’s theorem for Lévy processes for the special case of the Esscher transform. Here, we reformulate the result of example 9.5 and give a short proof (inspired by Eberlein and Papapantoleon 2005).

Proposition 9.7. Let $L = (L_t)_{t \geq 0}$ be a Lévy process with canonical decomposition (9.1) under $P$ and assume that $\mathbb{E}[\alpha L_t] < \infty$ for all $\alpha \in [-p, p]$, $p > 0$. Assume that $P \sim \tilde{P}$ with Radon-Nikodym derivative $\eta_T$, where
\begin{equation}
\eta_t = \frac{e^{\beta L_t} e^{\alpha L_t}}{\mathbb{E}[e^{\beta L_t}] \mathbb{E}[e^{\alpha L_t}]}
= \exp \left( \beta \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}} \alpha x (\mu^L - \nu^L)(ds, dx) \right)
- \left[ c \beta^2 + \int_{\mathbb{R}} (e^{\alpha x} - 1 - \alpha x) \nu(dx) \right] t,
\end{equation}
for $\beta \in \mathbb{R}$, $|\alpha| < p$. Then, $L$ remains a Lévy process under $\tilde{P}$, its Lévy triplet is $(\tilde{b}, c, f \nu)$, where $f(x) = e^{\alpha x}$ and its canonical decomposition is given by equations
\begin{equation}
L_t = \tilde{b} t + \sqrt{c} \tilde{W}_t + \int_0^t \int_{\mathbb{R}} x (\mu^L - \tilde{\nu}^L)(ds, dx),
\end{equation}
\begin{equation}
\tilde{b} = b + \beta c + \int_{\mathbb{R}} (e^{\alpha x} - 1) \nu(dx).
\end{equation}
Proof. Applying Theorem 25.17 in Sato (1999), the moment generating function \( M_{Lt} \) of \( L_t \) exists for \( u \in \mathbb{C} \) with \( \Re u \in [-p, p] \). We get
\[
\mathbb{E}[e^{zLt}] = \mathbb{E}[e^{zL_0}] e^{t(z + \beta)L_t} e^{zL_t} \mathbb{E}[e^{\alpha L_t}]^{-1} \mathbb{E}[e^{\beta L_t}]^{-1} \mathbb{E}[e^{\alpha L_t}]^{-1}
\]
\[
= \exp \left( t \left[ zb + \frac{(z + \beta)^2 c}{2} + \int_{\mathbb{R}} (e^{(z + \alpha)x} - 1 - (z + \alpha)x) \nu(dx) \right] - \frac{\beta^2 c}{2} \int_{\mathbb{R}} (e^{\alpha x} - 1) \nu(dx) \right)
\]
\[
= \exp \left( t \left[ zb + \frac{z^2 c}{2} + z \beta c + \int_{\mathbb{R}} e^{\alpha x} (e^{z x} - 1 - z x) \nu(dx) \right] + \int_{\mathbb{R}} z x (e^{\alpha x} - 1) \nu(dx) \right)
\]
\[
= \exp \left( t \left[ z (b + \beta c) + \int_{\mathbb{R}} x (e^{\alpha x} - 1) \nu(dx) \frac{z^2 c}{2} + \int_{\mathbb{R}} (e^{z x} - 1 - z x) e^{\alpha x} \nu(dx) \right] \right).
\]

The statement follows by proving that \( e^{\alpha x} \nu(dx) \) is a Lévy measure, i.e. \( \int_{\mathbb{R}} (1 \wedge x^2) e^{\alpha x} \nu(dx) < \infty \). It suffices to note that
\[
(9.10) \quad \int_{|x| \leq 1} x^2 e^{\alpha x} \nu(dx) \leq C \int_{|x| \leq 1} x^2 \nu(dx) < \infty,
\]
because \( \nu \) is a Lévy measure, where \( C \) is a constant, while the other part follows from the assumptions. \( \Box \)

10. Martingales and Lévy Processes

We give a condition for a Lévy process to be a martingale and discuss when the exponential of a Lévy process is a martingale.

**Proposition 10.1.** Let \( L_t = (L_t)_{t \geq 0} \) be a Lévy process with Lévy triplet \((b, c, \nu)\) and assume that \( \mathbb{E}|L_t| < \infty \). \( L_t \) is a martingale if and only if \( b = 0 \).

**Proof.** Follows from Theorem 5.2.1 in Applebaum (2004). \( \Box \)

**Proposition 10.2.** Let \( L_t = (L_t)_{t \geq 0} \) be a Lévy process, assume that \( \mathbb{E}[e^{\mu L_t}] < \infty, \mu \in \mathbb{R} \) and denote by \( \phi \) the cumulant (log-moment generating function) of \( L_t \). The process \( M_t = (M_t)_{t \geq 0}, \) defined as
\[
M_t = \frac{e^{\mu L_t}}{e^{\phi(u)}}
\]
is a martingale.

Proof. We have that \( E[e^{uL_t}] = e^{t\phi(u)} < \infty \), for all \( t \geq 0 \).

For \( 0 \leq s \leq t \), we can re-write \( M \) as

\[
M_t = e^{uL_s} e^{u(L_t - L_s)} - e^{s\phi(u)} e^{(t-s)\phi(u)} = M_s e^{(t-s)\phi(u)}.
\]

Using the fact that a Lévy process has stationary and independent increments, we get

\[
E[M_t \mid \mathcal{F}_s] = M_s \frac{e^{u(L_t - L_s)}}{e^{s\phi(u)} e^{(t-s)\phi(u)}} = M_s e^{(t-s)\phi(u)} e^{-(t-s)\phi(u)}
\]

\[
= M_s
\]

\[
\square
\]

The stochastic exponential \( \mathcal{E}(L) \) of a Lévy process \( L = (L_t)_{t \geq 0} \) is the solution \( Z \) of the stochastic differential equation

\[
dZ_t = Z_t - dL_t, \quad Z_0 = 1
\]

defined as

\[
(10.1) \quad \mathcal{E}(L)_t = \exp \left( L_t - \frac{ct}{2} \right) \prod_{0 \leq s \leq t} \left( 1 + \Delta L_s \right) e^{-\Delta L_s}.
\]

The stochastic exponential of a Lévy process that is a martingale is a local martingale (cf. Jacod and Shiryaev 2003, Theorem I.4.61) and indeed a martingale when working of a finite time horizon (cf. Kallsen 2000, Lemma 4.4).

### 11. Construction of Lévy processes

Three popular methods to construct a Lévy process are described below.

- **(C1):** Specifying a Lévy triplet: more specifically, whether there exists a Brownian component or not and what is the Lévy measure. Examples of Lévy process constructed this way include the standard Brownian motion, which has Lévy triplet \((0, 1, 0)\) and the Lévy jump-diffusion, which has Lévy triplet \((b, \sigma^2, \lambda F)\).

- **(C2):** Specifying an infinitely divisible random variable as the density of the increments at time scale 1 (i.e. \( L_1 \)). Examples of Lévy process constructed this way include the standard Brownian motion, where \( L_1 \sim \text{Normal}(0, 1) \) and the normal inverse Gaussian process, where \( L_1 \sim \text{NIG}(\alpha, \beta, \delta, \mu) \).

- **(C3):** Time-changing Brownian motion with an independent increasing Lévy process. Let \( W \) denote the standard Brownian motion; we can construct a Lévy process by “replacing” the (calendar) time \( t \) by an independent increasing Lévy process \( \tau \), therefore \( L_t := W_{\tau(t)} \), \( t \geq 0 \). The process \( \tau \) has the useful –in Finance– interpretation as “business time”. Models constructed this way include the standard Brownian motion, the normal inverse Gaussian process, where Brownian motion is time-changed with the inverse Gaussian process (hence its name) and the variance gamma process, where Brownian motion is time-changed with the gamma process.
Naturally, some processes can be constructed using more than one methods. Nevertheless, each method has some distinctive advantages which are very useful in applications. The advantages of specifying a triplet \((C1)\) are that the characteristic function and the pathwise properties are known and allows the construction of a rich variety of models; the drawbacks are that parameter estimation and simulation (in the infinite activity case) can be quite involved. The second method \((C2)\) allows the easy estimation and simulation of the process; on the contrary the structure of the paths might be unknown. The method of time-changes \((C3)\) allows for easy simulation, yet estimation might be quite difficult.

12. Simulation of Lévy processes

We shall briefly describe simulation methods for Lévy processes. We concentrate on finite activity Lévy processes (i.e. Lévy jump-diffusions) and some special cases of infinite activity Lévy processes, namely the normal inverse Gaussian and the variance gamma process. Here we do not discuss methods for simulation of random variables with known density; various simulation algorithms can be found in Devroye (1986) (also available online at http://jeff.cs.mcgill.ca/~luc/rnbookindex.html). Moreover, several speed-up methods for the Monte Carlo simulation of Lévy processes are discussed in Webber (2005).

12.1. Finite activity. Assume we want to simulate the Lévy jump-diffusion

\[
L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k
\]

where \(N_t \sim \text{Poisson}(\lambda t)\) and \(J \sim F(dx)\).

We can simulate a discretized trajectory of the Lévy jump-diffusion \(L\) at fixed time points \(t_1, \ldots, t_n\) as follows:

- simulate a standard normal variate
- transform it into a normal variate with variance \(\sigma \Delta t\), where \(\Delta t = t_i - t_{i-1}\) (denoted \(G_i\))
- simulate a Poisson random variate with parameter \(\lambda \Delta t\)
- simulate the law of jump sizes \(J\), i.e. simulate \(F(dx)\)
- if the Poisson variate is larger than zero, add the value of the jump.

The discretized trajectory is

\[
L_{t_i} = bt_i + \sum_{j=1}^{i} G_i + \sum_{k=1}^{N_t} J_k.
\]

12.2. Infinite activity. We will describe simulation methods for two very popular models in mathematical finance, the variance gamma and the normal inverse Gaussian process. These two processes can be easily simulated because they are time-changed Brownian motions; we follow Cont and Tankov (2003) closely. A general treatment of simulation methods for infinite activity Lévy processes can be found in Cont and Tankov (2003) and Schoutens (2003).
Assume we want to simulate a normal inverse Gaussian (NIG) process with parameters $\sigma, \theta, \kappa$; we can simulate a discretized trajectory at fixed time points $t_1, \ldots, t_n$ as follows:

- simulate $n$ independent inverse Gaussian variables $I_i$ with parameters $\lambda_i = (\Delta t)^{2\kappa}$ and $\mu_i = \Delta t$, where $\Delta t = t_i - t_{i-1}$, $i = 1, \ldots, n$
- simulate $n$ standard normal variables $G_i$
- set $\Delta L_i = \theta I_i + \sigma \sqrt{I_i} G_i$

The discretized trajectory is

$$L_{t_i} = \sum_{j=1}^{i} \Delta L_j.$$ 

Assume we want to simulate a variance gamma (VG) process with parameters $\sigma, \theta, \kappa$; we can simulate a discretized trajectory at fixed time points $t_1, \ldots, t_n$ as follows:

- simulate $n$ independent gamma variables $\Gamma_i$ with parameter $\frac{\Delta t}{\kappa}$ where $\Delta t = t_i - t_{i-1}, i = 1, \ldots, n$
- set $\Gamma_i = \kappa \Gamma_i$
- simulate $n$ standard normal variables $G_i$
- set $\Delta L_i = \theta \Gamma_i + \sigma \sqrt{\Gamma_i} G_i$

The discretized trajectory is

$$L_{t_i} = \sum_{j=1}^{i} \Delta L_j.$$ 

13. Asset price model

We describe an asset price model driven by a Lévy process, both under the “real” and under the “risk-neutral” measure.

13.1. Real-world measure. Under the real-world measure, we model the asset price process as the exponential of a Lévy process, that is

$$S_t = S_0 \exp \{L_t\};$$ (13.1)

here, $L$ is the Lévy process whose infinitely divisible distribution has been estimated from the data set available for the particular asset. Therefore, the log-returns of the model have independent and stationary increments, which are distributed—along time intervals of specific length, for example 1—according to an infinitely divisible distribution $X$, i.e. $L_1 \overset{d}{=} X$.

Naturally, the path properties of the process $L$ carry over to $S$; if, for example, $L$ is a pure-jump Lévy process, then $S$ is also a pure-jump process. This fact allows us to capture, up to a certain extend, the microstructure of price fluctuations, even on an intraday time scale.

The fact that the price process is driven by a Lévy process, makes the market, in general, incomplete; the only exceptions are the markets driven by the Normal (Black-Scholes model) and Poisson distributions. Therefore, there exists a large set of equivalent martingale measures, i.e. candidate measures for risk-neutral valuation.
Eberlein and Jacod (1997) provide a thorough analysis and characterization of the set of equivalent martingale measures; moreover, they prove that the range of, e.g. call, option prices under all possible equivalent martingale measures spans the whole no-arbitrage interval (i.e $[(S_0 - Ke^{-rT})^+, S_0]$ for a European call option with strike $K$). In addition, Selivanov (2005) discusses the existence and uniqueness of martingale measures for exponential Lévy models in finite and infinite time horizon and for various specifications of the no-arbitrage condition.

The Lévy market can be completed using particular assets, such as moment derivatives (e.g. variance swaps), and then there exists a unique equivalent martingale measure; see Corcuera, Nualart, and Schoutens (2005a, 2005b). For example, if an asset is driven by a Lévy jump-diffusion

\[ L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k \]

where $F(dx) \equiv 1$, then the market can be completed using only variance swaps on this asset.

The process $(S_t)_{t \geq 0}$ is the solution of the stochastic differential equation

\[ dS_t = S_t \left( dL_t + \frac{1}{2} \sigma^2 dt + e^{\Delta L_t} - 1 - \Delta L_t \right), \]

where $\Delta L_t = L_t - L_{t-}$ is the jump of $L$ at time $t$. We could also specify $S$ as the solution to the following SDE

\[ dS_t = S_t \, dL_t, \]

whose solution is the stochastic exponential $S_t = S_0 \mathcal{E}(L_t)$. The second approach is unfavorable for financial applications, because (a) the asset price can take negative values, unless jumps are restricted to be larger than $-1$ (i.e. supp($\nu$) $\subset [-1, \infty)$) and (b) the distribution of log-returns is not known. In the special case of the Black-Scholes model, the two approaches coincide.

**Remark 13.1.** For the connection between the natural and stochastic exponential for Lévy processes, we refer to Lemma A.8 in Goll and Kallsen (2000).

13.2. **Risk-neutral measure.** Under the risk neutral measure, denoted by $P$, we will model the asset price process as an exponential Lévy process

\[ S_t = S_0 \exp L_t \]

where the Lévy process $L$ must satisfy Assumptions (M) and (EM).

**Assumption (EM).** We assume that the Lévy process $L$ has a finite first exponential moment, i.e.

\[ \mathbb{E}[e^{L_t}] < \infty. \]

There are various ways to choose the martingale measure such that it is equivalent to the real-world measure. We refer to Goll and Rüschendorf (2001) for a unified exposition – in terms of $f$-divergences – of the different methods for selecting an equivalent martingale measure (EMM). Note that, some of the proposed methods to choose an EMM preserve the Lévy property
of log-returns; examples are the Esscher transformation and the minimal entropy martingale measure (cf. Esche and Schweizer 2005).

The market practice is to consider the choice of the martingale measure as the result of a calibration to market data of vanilla options. Hakala and Wystup (2002) describe the calibration procedure in detail; moreover, Cont and Tankov (2004, 2005) and Belomestny and Reiß (2005) present numerically stable calibration methods for Lévy driven models.

Because we have assumed that $P$ is a risk neutral measure, the asset price has mean rate of return $\mu \triangleq r - \delta$ and the discounted and re-invested process $(e^{(r-\delta)\frac{1}{2}}S_t)_{t \geq 0}$, is a martingale under $P$. Here $r \geq 0$ is the (domestic) risk-free interest rate, $\delta \geq 0$ the continuous dividend yield (or foreign interest rate) of the asset.

The process $L$ has the canonical decomposition

$$ L_t = bt + \sigma W_t + \int_0^t \int_\mathbb{R} x(\mu^L - \nu^L)(ds, dx) $$

and hence, the drift term $b$ equals the expectation of $L_1$ and can be written as

$$ b = r - \delta - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - 1 - x) \nu(dx). $$

where $\sigma \geq 0$ the diffusion coefficient and $W$ a standard Brownian motion under $P$. $\mu^L$ is the random measure of jumps of the process $L$ and $\nu^L(dt, dx) = \nu(dx)dt$ is the compensator of the jump measure $\mu^L$, where $\nu$ is the Lévy measure of $L_1$.

14. Popular models

14.1. Black-Scholes. The most famous asset price model based on a Lévy process is that of Samuelson (1965), Black and Scholes (1973) and Merton (1973). The log-returns are normally distributed with mean $\mu$ and variance $\sigma^2$, i.e. $L_1 \sim \text{Normal}(\mu, \sigma^2)$ and the density is

$$ f_{L_1}(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[ -\frac{(x - \mu)^2}{2\sigma^2} \right]. $$

The characteristic function is

$$ \varphi_{L_1}(u) = \exp\left[ i\mu u - \frac{\sigma^2 u^2}{2} \right], $$

the first and second moments are

$$ \text{E}[L_1] = \mu, \quad \text{Var}[L_1] = \sigma^2, $$

while the skewness and kurtosis are

$$ \text{skew}[L_1] = 0, \quad \text{kurt}[L_1] = 3. $$

The canonical decomposition of $L$ is

$$ L_t = \mu t + \sigma W_t $$

and the Lévy triplet is $(\mu, \sigma^2, 0)$. 
14.2. **Merton.** Merton (1976) was the one of the first to use a discontinuous price process to model asset returns. The canonical decomposition of the driving process is

\[ L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k \]

where \( J_k \sim \text{Normal}(\mu_J, \sigma_J^2), \ k = 1, \ldots \), hence the distribution of the jump size has the density

\[ f_J(x) = \frac{1}{\sigma_J \sqrt{2\pi}} \exp \left[ -\frac{(x - \mu_J)^2}{2\sigma_J^2} \right]. \]

The characteristic function of \( L_1 \) is

\[ \varphi_{L_1}(u) = \exp \left[ i\mu u - \frac{\sigma^2 u^2}{2} + \lambda \left( e^{i\mu u - \sigma^2 u^2/2} - 1 \right) \right], \]

and the Lévy triplet is \((\mu, \sigma^2, \lambda \cdot f_J)\).

The density of \( L_1 \) is not known in closed form, while the first two moments are

\[ E[L_1] = \mu + \lambda \mu_J \quad \text{and} \quad \text{Var}[L_1] = \sigma^2 + \lambda \mu_J^2 + \lambda \sigma_J^2. \]

14.3. **Kou.** Kou (2002) proposed a jump-diffusion model similar to Merton’s, where the jump size is double-exponentially distributed. Therefore, the canonical decomposition of the driving process is

\[ L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k \]

where \( J_k \sim \text{DbExpo}(p, \theta_1, \theta_2), \ k = 1, \ldots \), hence the distribution of the jump size has the density

\[ f_J(x) = \begin{cases} p \theta_1 e^{-\theta_1 x} & \text{if } x \leq 0 \\ (1-p) \theta_2 e^{\theta_2 x} & \text{if } x > 0 \end{cases}. \]

The characteristic function of \( L_1 \) is

\[ \varphi_{L_1}(u) = \exp \left[ i\mu u - \frac{\sigma^2 u^2}{2} + \lambda \left( \frac{p \theta_1}{\theta_1 - iu} - \frac{(1-p) \theta_2}{\theta_2 + iu} - 1 \right) \right], \]

and the Lévy triplet is \((\mu, \sigma^2, \lambda \cdot f_J)\).

The density of \( L_1 \) is not known in closed form, while the first two moments are

\[ E[L_1] = \mu + \frac{\lambda p}{\theta_1} - \frac{\lambda (1-p)}{\theta_2} \quad \text{and} \quad \text{Var}[L_1] = \sigma^2 + \frac{\lambda p}{\theta_1^2} + \frac{\lambda (1-p)}{\theta_2^2}. \]

14.4. **Generalized Hyperbolic.** The generalized hyperbolic model was introduced by Eberlein and Prause (2002) following the works on the hyperbolic model by Eberlein and Keller (1995); the class of hyperbolic distributions was invented by O. E. Barndorff-Nielsen in relation to the so-called “sand project” (cf. Barndorff-Nielsen 1977). The increments of time length
1 follow a generalized hyperbolic distribution with parameters $\alpha, \beta, \delta, \mu, \lambda$, i.e. $L_1 \sim \text{GH}(\alpha, \beta, \delta, \mu, \lambda)$ and the density is

$$f_{GH}(x) = c(\lambda, \alpha, \beta, \delta, \mu, \lambda) \left\{ \begin{array}{l} \alpha^2 (x - \mu)^{\lambda-\frac{1}{2}} \times K_{\lambda-\frac{1}{2}}(\alpha \sqrt{\delta^2 + (x - \mu)^2}) \exp(\beta(x - \mu)), \end{array} \right.$$ where

$$c(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi} \alpha^\lambda K_{\lambda-\frac{1}{2}}(\delta \sqrt{\alpha^2 - \beta^2})}$$

and $K_{\lambda}$ denotes the Bessel function of the third kind with index $\lambda$ (cf. Abramowitz and Stegun 1968). Parameter $\alpha > 0$ determines the shape, $0 \leq |\beta| < \alpha$ determines the skewness, $\mu \in \mathbb{R}$ the location and $\delta > 0$ is a scaling parameter. The last parameter, $\lambda \in \mathbb{R}$ affects the heaviness of the tails and allows us to navigate through different subclasses. For example, for $\lambda = 1$ we get the hyperbolic distribution and for $\lambda = -\frac{1}{2}$ we get the normal inverse Gaussian (NIG).

The characteristic function of the GH distribution is

$$\varphi_{GH}(u) = e^{iu\mu} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\lambda/2} \frac{K_{\lambda}(\delta \sqrt{\alpha^2 - (\beta + iu)^2})}{K_{\lambda}(\delta \sqrt{\alpha^2 - \beta^2})},$$

while the first and second moments are

$$E[L_1] = \mu + \frac{\beta \delta^2 K_{\lambda+1}(\zeta)}{\zeta K_{\lambda}(\zeta)}$$

and

$$\text{Var}[L_1] = \frac{\delta^2 K_{\lambda+1}(\zeta)}{\zeta K_{\lambda}(\zeta)} + \frac{\beta^2 \delta^4}{\zeta^2} \left( \frac{K_{\lambda+2}(\zeta)}{K_{\lambda}(\zeta)} - \frac{K_{\lambda+1}^2(\zeta)}{K_{\lambda}^2(\zeta)} \right),$$

where $\zeta = \delta \sqrt{\alpha^2 - \beta^2}$.

The canonical decomposition of a Lévy process driven by a generalized hyperbolic distribution (i.e. $L_1 \sim \text{GH}$) is

$$L_t = tE[GH] + \int_0^t \int \mu^L - \nu^GH (ds, dx),$$

and the Lévy triplet is $(E[GH], 0, \nu^GH)$; the Lévy measure of the GH distribution is known, but has a quite complicated expression, see Raible (2000, section 2.4.1)

The GH distribution contains as special or limiting cases several known distributions, including the normal, exponential, gamma, variance gamma, hyperbolic and normal inverse Gaussian distributions; see Eberlein and v. Hammerstein (2004).

14.5. **Normal Inverse Gaussian.** The normal inverse Gaussian distribution is a special case of the GH for $\lambda = -\frac{1}{2}$; it was introduced to finance in
Barndorff-Nielsen (1997). The density is
\[
f_{NIG}(x) = \frac{\alpha}{\pi} \exp\left(\delta \sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\right) \frac{K_1(\alpha \delta \sqrt{1 + (\frac{x-\mu}{\beta})^2})}{\sqrt{1 + (\frac{x-\mu}{\beta})^2}},
\]
while the characteristic function has the simplified form
\[
\varphi_{NIG}(u) = e^{iu\mu} \frac{\exp(\delta \sqrt{\alpha^2 - \beta^2})}{\exp(\delta \sqrt{\alpha^2 - (\beta + iu)^2})}.
\]
The first and second moments of the NIG distribution are
\[
E[L_1] = \mu + \frac{\beta \delta}{\sqrt{\alpha^2 - \beta^2}} \quad \text{and} \quad \text{Var}[L_1] = \frac{\delta}{\sqrt{\alpha^2 - \beta^2}} + \frac{\beta^2 \delta}{(\sqrt{\alpha^2 - \beta^2})^3},
\]
and similarly to the GH, the canonical decomposition is
\[
L_t = tE[NIG] + \int_0^t \int \mu_L s - \nu_{NIG}(ds, dx),
\]
where now the Lévy measure has the form
\[
\nu_{NIG}(dx) = e^{\beta x} \frac{\delta \alpha}{\pi |x|} K_1(\alpha |x|) dx.
\]
The NIG is the only subclass of the GH that is closed under convolution, i.e. if \( X \sim \text{NIG}(\alpha, \beta, \delta_1, \mu_1) \) and \( Y \sim \text{NIG}(\alpha, \beta, \delta_2, \mu_2) \) and \( X \) is independent of \( Y \), then
\[
X + Y \sim \text{NIG}(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2).
\]
Therefore, if we estimate the returns distribution at some time scale, then we know it –in closed form– for all time scales.

15. Pricing European options

We review the idea of Sebastian Raible for the valuation of European options using Laplace transforms; see chapter 3 and appendix B.1 in Raible (2000). We assume that the following conditions are in force.

15.1. Summary of assumptions. The necessary assumptions regarding the return distribution of the asset are:

(D1): Assume that \( \varphi_{L_T}(z) \), the extended characteristic function of \( L_T \), exists for all \( z \in \mathbb{C} \) with \( \Im z \in I_1 \supset [0, 1] \).

(D2): Assume that \( P_{L_T} \), the distribution of \( L_T \), is absolutely continuous w.r.t. the Lebesgue measure \( \lambda \) with density \( \rho \).

The necessary assumptions regarding the payoff function are:

(P1): Consider a European-style payoff function \( f(S_T) \) that is integrable.

(P2): Assume that \( x \mapsto e^{-Rx}|f(e^{-x})| \) is bounded and integrable for all \( R \in I_2 \subset \mathbb{R} \).

Finally, we need the following assumption involving both the payoff function and the return distribution of the asset:

(B1): Assume that \( I_1 \cap I_2 \neq \emptyset \).
15.2. Raible’s method.

**Definition 15.1.** Let $\mathcal{L}_h(z)$ denote the bilateral Laplace transform of a function $h$ at $z \in \mathbb{C}$, i.e. let

$$\mathcal{L}_h(z) := \int_{\mathbb{R}} e^{-zx} h(x) dx.$$ 

The next Theorem gives an explicit expression for the value of an option with payoff function $f(\cdot)$ and driving process $L$.

**Theorem 15.2.** Assume that the above mentioned conditions are in force and let $g(x) := f(e^{-x})$ denote the modified payoff function of an option with payoff $f(x)$ at time $T$. Choose an $R \in I_1 \cap I_2$. Letting $V(\zeta)$ denote the price of this option, as a function of $\zeta := -\log S_0$, we have

$$V(\zeta) = e^{\zeta R - rT} \frac{e^{i\zeta(R + iu)}}{2\pi} \int_{\mathbb{R}} e^{iu \zeta} \mathcal{L}_g(R + iu) \varphi_{LT}(iR - u) du,$$

whenever the integral on the r.h.s. exists (at least as a Cauchy principal value).

**Proof.** According to arbitrage pricing, the value of an option is equal to its discounted expected payoff under the risk-neutral measure $P$. We get

$$V = e^{-rT} \mathbb{E}[f(S_T)]$$

$$= e^{-rT} \int_{\Omega} f(S_T) dP$$

$$= e^{-rT} \int_{\mathbb{R}} f(S_0e^x) dP_{LT}(x)$$

$$= e^{-rT} \int_{\mathbb{R}} f(S_0e^x) \rho(x) dx$$

because $P_{LT}$ is absolutely continuous with respect to the Lebesgue measure (by D2). Define the function $g(x) = f(e^{-x})$ and let $\zeta = -\log S_0$, then

$$V = e^{-rT} \int_{\mathbb{R}} g(\zeta - x) \rho(x) dx = e^{-rT} (g * \rho)(\zeta)$$

which is a convolution of $g$ with $\rho$ at the point $\zeta$, multiplied by the discount factor.

The idea now is to apply a Laplace transform on both sides of (15.2) and take advantage of the fact that the Laplace transform of a convolution equals the product of the Laplace transforms of the factors. The resulting Laplace transforms are easier to calculate analytically. Finally, we can invert the Laplace transforms to recover the option value.

For the functions $g$ and $\rho$ we have that $x \mapsto e^{-Rx} |g(x)|$ is bounded,

$$\int_{\mathbb{R}} e^{-Rx} |g(x)| dx < \infty$$
for $R \in I_2$ (by P2) and

$$\int_{\mathbb{R}} e^{-Rx} \rho(x) dx = \varphi_{L_T}(iR) < \infty$$

for $R \in I_1$ (by D1). Therefore, for $R \in I_1 \cap I_2$ (use B1) the prerequisites of Theorem B.2 in Raible (2000) are met and we get that, the convolution in (15.2) is continuous and absolutely convergent. Applying Theorem B.2, we get, for $z = R + iu, u \in \mathbb{R}$

$$\mathcal{L}_V(z) = e^{-rT} \int_{\mathbb{R}} e^{-zx}(g * \rho)(x) dx$$

$$= e^{-rT} \int_{\mathbb{R}} e^{-zx} g(x) dx \int_{\mathbb{R}} e^{-zx} \rho(x) dx$$

$$= e^{-rT} \mathcal{L}_g(z) \mathcal{L}_\rho(z).$$

Moreover, from the same Theorem we have that $\zeta \mapsto V(\zeta)$ is continuous and

$$\int_{\mathbb{R}} e^{-R|\zeta|} |V(\zeta)| d\zeta < \infty.$$

Hence, the prerequisites of Theorem B.3 in Raible (2000) are satisfied and we can invert this Laplace transform to recover the option value.

$$V(\zeta) = \frac{1}{2\pi i} \int_{C}^{R+i\infty}_{R-i\infty} e^{\zeta z} \mathcal{L}_V(z) dz$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(R+iu)} \mathcal{L}_V(R + iu) du$$

$$= \frac{e^{iR}}{2\pi} \int_{\mathbb{R}} e^{iu} e^{-rT} \mathcal{L}_g(R + iu) \mathcal{L}_\rho(R + iu) du$$

$$= \frac{e^{-rT}e^{iR}}{2\pi} \int_{\mathbb{R}} e^{iu} \mathcal{L}_g(R + iu) \varphi_{L_T}(iR - u) du.$$

\[\square\]

15.3. Examples of options.

**Example 15.3** (Plain vanilla option). A European plain vanilla call option pays off $f(S_T) = (S_T - K)^+$, for some strike price $K$. The Laplace transform of its modified payoff function $g$ (normalized for strike $K = 1$) is

(15.3) \hspace{1cm} \mathcal{L}_g(z) = \frac{1}{z(z + 1)}

for $z \in \mathbb{C}$ with $\Re z = R \in I_2 = (-\infty, -1)$.

Similarly, for a European plain vanilla put option that pays off $f(S_T) = (K - S_T)^+$, the Laplace transform of its modified payoff function $g$ (normalized for strike $K = 1$) is given by (15.3) for $z \in \mathbb{C}$ with $\Re z = R \in I_2 = (0, \infty)$. 

Example 15.4 (Digital option). A European digital call option pays off \( f(S_T) = 1{\{S_T > K}\} \). The Laplace transform of its modified payoff function \( g \) is

\[
\mathcal{L}_g(z) = -\frac{1}{z} \left( \frac{K}{S_0} \right)^z
\]

for \( z \in \mathbb{C} \) with \( \Re z = R \in I_2 = (-\infty, 0) \).

Similarly, for a European digital put option that pays off \( f(S_T) = 1{\{S_T < K\}} \), the Laplace transform of its modified payoff function \( g \) is

\[
\mathcal{L}_g(z) = \frac{1}{z} \left( \frac{K}{S_0} \right)^z
\]

for \( z \in \mathbb{C} \) with \( \Re z = R \in I_2 = (0, \infty) \).

15.4. Examples of distributions.

Example 15.5 (Normal). The Normal distribution has a (known) Lebesgue density and its moment generating function \( M_N(u) \) exists for all \( u \in \mathbb{R} \), therefore we have that \( I_1 = \mathbb{R} \).

Example 15.6 (Generalized Hyperbolic). The Generalized Hyperbolic distribution has a (known) Lebesgue density and its moment generating function \( M_{GH}(u) \) exists for all \( u \in (-\alpha - \beta, \alpha - \beta) \), therefore we have that \( I_1 = (-\alpha - \beta, \alpha - \beta) \).

16. Empirical evidence

Lévy processes provide a framework that can easily capture the empirical observations both under the “real world” and the “risk-neutral” measure. We provide here some indicative examples.

Under the “real world” measure, Lévy processes can be generated by distributions that are flexible enough to capture the observed fat-tailed and skewed (leptokurtic) behavior of asset returns. One such class of distributions is the class of generalized hyperbolic distributions (cf. section 14.4). In Figure 16.11, various densities of generalized hyperbolic distributions and a comparison of a generalized hyperbolic and normal density are plotted.

A typical example of the behavior of asset returns can be seen in Figures 1.2 and 16.12. The fitted normal distribution has lower peak, fatter flanks.

Figure 16.11. Densities of hyperbolic (red), NIG (blue) and hyperboloid (green) distributions (left). Comparison of the GH (red) and Normal (blue) distributions.
and lighter tails than the empirical distribution; this means that, in reality, tiny and large price movements occur more frequently, and small and medium size movements occur less frequently, than predicted by the normal distribution. On the other hand, the generalized hyperbolic distribution gives a very good statistical fit of the empirical distribution; this is further verified by the corresponding Q-Q plot.

Under the “risk-neutral” measure, the flexibility of the generating distributions allows to accurately capture the shape of the implied volatility smile and to a lesser extend the shape of the whole volatility surface.

17. RELATED LITERATURE

The following articles are recommended for further reading.


The following books are recommended for further reading.


APPENDIX A. POISSON RANDOM VARIABLES AND PROCESSES

**Definition A.1.** Let \( X \) be a Poisson distributed random variable with parameter \( \lambda \in \mathbb{R}_+ \). Then, for \( n \in \mathbb{N} \) the probability distribution is

\[
P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}
\]

and the first two centered moments are

\[
E[X] = \lambda \quad \text{and} \quad \text{Var}[X] = \lambda.
\]
Definition A.2. An adapted càdlàg process $N_t : \Omega \times \mathbb{R}_+ \to \mathbb{N} \cup \{0\}$ is called a Poisson process if

1. $N_0 = 0$,
2. $N_t - N_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t < T$,
3. $N_t - N_s$ is Poisson distributed with parameter $\lambda (t - s)$ for any $0 \leq s < t < T$.

Then, $\lambda \geq 0$ is called the intensity of the Poisson process.

Definition A.3. Let $N$ be a Poisson process with parameter $\lambda$. We shall call the process $\overline{N}_t : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ where

\begin{equation}
\overline{N}_t := N_t - \lambda t
\end{equation}

a compensated Poisson process.

A simulated path of a Poisson and a compensated Poisson process can be seen in Figure A.13.

![Plots of the Poisson (left) and compensated Poisson process.](image)

Figure A.13. Plots of the Poisson (left) and compensated Poisson process.

Proposition A.4. The compensated Poisson process defined by (A.1) is a martingale.

Proof. We have

1. The process $\overline{N}$ is adapted to the filtration because $N$ is adapted (by definition),
2. $\mathbb{E}[|\overline{N}_t|] < \infty$ because $\mathbb{E}[|N_t|] < \infty$,
3. Let $0 \leq s < t < T$, then

\[ \mathbb{E}[\overline{N}_t | \mathcal{F}_s] = \mathbb{E}[N_t - \lambda t | \mathcal{F}_s] = \mathbb{E}[N_t - (N_t - N_s) | \mathcal{F}_s] - \lambda (s - (t - s)) = N_s - \mathbb{E}[(N_t - N_s) | \mathcal{F}_s] - \lambda (s - (t - s)) = N_s - \lambda s = \overline{N}_s. \]
Remark A.5. The characteristic functions of the Poisson and compensated Poisson processes are respectively

\[ \mathbb{E}[e^{iuN_t}] = \exp \left( \lambda t(e^{iu} - 1) \right) \]

and

\[ \mathbb{E}[e^{iu\tilde{N}_t}] = \exp \left( \lambda t(e^{iu} - 1 - iu) \right). \]

Appendix B. Compound Poisson random variables

Let \( N \) be a Poisson distributed random variable with parameter \( \lambda > 0 \) and \( J = (J_k)_{k \geq 1} \) an i.i.d. sequence of random variable with law \( F \). Then, by conditioning on the number of jumps and using independence, we have that the characteristic function of a compound Poisson distributed random variable is

\[ \mathbb{E}[e^{iu \sum_{k=1}^{N} J_k}] = \sum_{n \geq 0} \mathbb{E}[e^{iu \sum_{k=1}^{n} J_k} | N = n] P(N = n) \]

\[ = \sum_{n \geq 0} \mathbb{E}[e^{iu \sum_{k=1}^{n} J_k}] e^{-\lambda \lambda^n / n!} \]

\[ = \sum_{n \geq 0} \left( \int_{\mathbb{R}} e^{iux} F(dx) \right)^n e^{-\lambda \lambda^n / n!} \]

\[ = \exp \left( \lambda \int_{\mathbb{R}} (e^{iux} - 1) F(dx) \right). \]

Appendix C. Notation

\( d \) equality in law
\( a \wedge b = \min\{a, b\} \)
\( a \vee b = \max\{a, b\} \)

Let \( \mathbb{C} \ni z = x + iy, x, y \in \mathbb{R} \). Then \( \Re z = x \) and \( \Im z = y \).

\[ \mathbb{I}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \]

Appendix D. Datasets

The EUR/USD implied volatility data are from 5 November 2001. The spot price was 0.93, the domestic rate (USD) 5% and the foreign rate (EUR) 4%. The data are available at

http://www.mathfinance.de/FF/sampleinputdata.txt.


References


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