Bridging the Gap between Representative-Agent and Heterogeneous-Agent Models

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Understanding the differences and similarities between heterogeneous-agent models and representative-agent models has been at the forefront of macroeconomics for the last twenty years. While it is known that perfect macroeconomic aggregation can fail for even trivial reasons, this does not preclude use of representative-agent abstraction if the divergence implied for aggregate behavior is small. For instance, in a seminal paper on heterogeneity in macroeconomic models, Krusell and Smith (1998), finds that even in the face of idiosyncratic risk with heterogeneous income, wealth, and preferences, a model might only keep track of a "representative budget" with little loss in information, and that a heterogeneous-agent model might attain "approximate aggregation," in which it behaves substantially similarly to a representative agent model, in spite of a breakdown in perfect aggregation.

However, other papers find that modeling heterogeneity can cause substantial deviations in behavior, as in Chang and Kim (2007, 2014) and Takahashi (2014). This paper provides a bridge for understanding where representative-agent and heterogeneous-agent models might diverge. By quadraticizing univariate outcome response to a parameter change for both heterogeneous-agent and representative-agent models, I obtain a closed formula that relates differences to five causes: (1) the covariance of linear outcome responsiveness with a parameter change, (2) the variance of the parameter change, weighted by the mean quadratic responsiveness to the change, (3) the covariance of quadratic outcome responsiveness with a parameter change (4) the co-skewness of linear responsiveness with tax change, and (5) initial miscalibration in responsiveness.

My formula helps shed light on the conditions under which approximate aggregation can fail. I apply my model to understand four important macroeconomic models: the approximate aggregation of Krusell and Smith (1998), the failure of aggregation in Chang and Kim (2007) and subsequent papers, and two simple CGE models inspired by Prescott (2004) and Kaplan et al. (2014) to analyze labor income tax reform, and the marginal propensity to consume, respectively. In each, the formula I derive categorizes the similarities and divergences. In the case of Krusell and Smith (1998), the combination of homogeneous linear savings combined with no significant nonlinearities in savings precludes differences in, for instance, the response of the economy's savings rate to a shock in TFP. Even in the case of Kaplan et al. (2014), while there is significant convexity in individual agent's problems, these convexities largely "average out", allowing for a relatively accurate representative agent approximation. In the case of Chang and Kim (2007) and significant (and heterogeneous) nonlinearities are produced. In the case of tax reform, I show that some functional forms do not have enough degrees of freedom to be properly calibrated to both the level and the responsiveness, and that in such a case, calibration to the level of labor supply may cause significant misstatements. Even when it is not, covariance between responsiveness and tax changes can cause a dramatic overstatement of labor hours changes. Taken together, these results suggest a diagnostic role for my model, and suggests that covariance between responsiveness and an economic shock is an unambiguous threat to representative models, while nonlinearities in responsiveness are a concern when they are not evenly distributed among the population.

1 Literature Review

2 Model

Many models and analyses focus on a small subset of outcomes, such as the response of aggregate labor hours to a productivity shock or a tax change. The typical model takes in a set of preferences ($\mathcal{U}$), budget constraints ($\mathcal{I}$), laws of motion (Φ), and market-clearing conditions ($\mathcal{P}$) and solves for an agent's, or set of agent's, policy functions $\mathcal{G}$. For clarity of exposition, assume that the covariate of interest is the change in aggregate labor hours, denoted ΔL, and the parameter change of interest is a marginal tax rate change, $\Delta\tau$. I note that any outcome and stimulus variables can be exchanged for ΔL and $\Delta\tau$. Then a simple representative agent model $\mathcal{M}^R$ has the change in labor hours:

$$\Delta L^R = G(\mathcal{M}(\mathcal{U}, \mathcal{I}, \Phi, \mathcal{P}), \Delta\tau)$$

Similarly, a simple heterogeneous agent model $\mathcal{M}^H$ has the change in labor hours:

$$\Delta L^H = \sum_{i=1}^{N} \Delta L_i = \sum_{i=1}^{N} \left(G(\mathcal{M}(\mathcal{U}, \mathcal{I}, \Phi, \mathcal{P}), \Delta\tau_i)\right)$$

Labor hours for an individual agent in a heterogeneous economy can be approximated by quadraticizing the response to a change in taxation:

$$\Delta L_i^H = \beta_i^H \Delta\tau_i^H + \gamma_i^H \left(\Delta\tau_i^H\right)^2 \tag{1}$$

where β_i^H and γ_i^H distinguish individual responsiveness and nonlinearities to marginal tax rate changes in the heterogeneous agent model. A representative agent model may approximate this by taking the same model of individuals (assumed to be correct here) and applying it to one "representative" individual[1]

$$\overline{\Delta L} = \overline{\beta} \cdot \overline{\Delta\tau} + \overline{\gamma} \left(\overline{\Delta\tau}\right)^2 \tag{2}$$

There are many potential choices for $\overline{\beta}$, $\overline{\gamma}$, and $\overline{\Delta\tau}$. We first examine the obvious choices, in which each is replaced by its sample mean:

$$\overline{\beta} = \frac{1}{N}\sum_{i=1}^{N}\beta_i \qquad \overline{\gamma} = \frac{1}{N}\sum_{i=1}^{N}\gamma_i \qquad \overline{\Delta\tau} = \frac{1}{N}\sum_{i=1}^{N}\Delta\tau_i$$

In such a case, the difference between the average heterogeneous agent response, derived by the mean of all agents in equation 1 and the representative agent's response, summarized by equation 2, can be solved for in closed form.

To do so, I note that there are several convenient ways to rewrite the sample means of $(\Delta\tau_i)^2$,

[1] It is important to note here that the representative agent's squared term squares the same term that $\overline{\beta}$ multiplies: an ordinary taylor approximation might instead choose some more representative $\overline{(\Delta\tau)^2} \neq \left(\overline{\Delta\tau}\right)^2$ for the squared term. In most models, however, there is only one $\Delta\tau$ input parameter, and the curvature is dictated by the model's quadratic reaction to $\Delta\tau$, rather than having two separate parameters. For instance, the representative budget constraint:

$$wL(1-\tau) + \nu = c$$

allows for only one τ parameter, with no explicitly separate linear and quadratic tax entry.

$\beta_i \Delta\tau_i$, and $\gamma_i \Delta\tau_i$ using the sample means of $\Delta\tau$, β, γ, as well as the variances, covariances, and co-skewness of the terms[2]. Specifically, it can be shown that:

$$\frac{1}{N}\sum_{i=1}^{N}\beta_i\Delta\tau_i = \overline{\beta}\cdot\overline{\Delta\tau} + Cov(\beta_i, \Delta\tau_i)$$
$$\frac{1}{N}\sum_{i=1}^{N}\gamma_i\Delta\tau_i^2 = \overline{\gamma}\cdot\overline{\Delta\tau}^2 + 2\overline{\Delta\tau}Cov(\gamma_i, \Delta\tau_i) + \overline{\gamma}Var(\Delta\tau_i) + S(\gamma_i, \Delta\tau_i, \Delta\tau_i)$$

Noting that the first term in each of the above approximations is a term the calibrated representative agent's response, we can write the heterogeneous response is the representative agent's response plus four additional terms:

$$\frac{1}{N}\sum_{i=1}^{N}\Delta L_i = \overline{\Delta L}^R + Cov(\beta_i, \Delta\tau_i) + \overline{\gamma}Var(\Delta\tau_i) + S(\gamma_i, \Delta\tau_i, \Delta\tau_i) + 2\overline{\Delta\tau}Cov(\gamma_i, \Delta\tau_i) \tag{3}$$

Moreover, miscalibration of responsiveness may be introduced as a fifth reason for divergence, noting that:

$$\overline{\Delta L} - \overline{\Delta L}^{MC} = (\overline{\beta} - \overline{\beta}^{MC})\Delta\tau + (\overline{\gamma} - \overline{\gamma}^{MC})(\overline{\Delta\tau})^2 \tag{4}$$

As I discuss, because many representative agent models are calibrated to levels, rather than elasticities, while heterogeneous agent models often have the degrees of freedom to match both, this fifth source is potentially important, and highlights that the use of flexible functional forms for the representative agent may be necessary to match a heterogeneous-agent model with simple functional forms.

Table 2 denotes the misstatement terms relevant in a properly-calibrated representative agent model, depending on whether or not there is heterogeneity in linear responses, heterogeneity in

[2] While co-skewness is ordinarily defined as the third mixed moment,

$$S(\gamma_i, \Delta\tau_i, \Delta\tau_i) = \frac{1}{N}\sum_{i=1}^{N}\frac{((\Delta\tau_i - \overline{\Delta\tau})^2(\gamma_i - \overline{\gamma}))}{Var(\Delta\tau_i)\sqrt{Var(\gamma_i)}}$$

It will instead be convenient to define it as the sum of the product of the squared deviations of τ_i and the deviations of β_i from their respective means. So, I write coskewness as:

$$S(\gamma_i, \Delta\tau_i, \Delta\tau_i) = \frac{1}{N}\sum_{i=1}^{N}((\Delta\tau_i - \overline{\Delta\tau})^2(\gamma_i - \overline{\gamma}))$$

quadratic responses, and heterogeneity in taxes (or the relevant stimulus variable). It suggests that heterogeneity in the tax change is necessary to generate a divergence between the two types of models. This follows straightforwardly from the fact that the heterogeneous-agent model can be properly stated as a representative agent model. It also helps clarify in what situations we should be concerned about heterogeneity, and what types of heterogeneity.

Summary: Representative Agent Misstatement of Aggregate Response

Heterogeneous Linear Response	Heterogeneous Quadratic Response	Heterogeneous Taxes	Rep Agent Misstatement
0	No term	1	0
0	0	1	$Var(\tau_i)$
0	1	0	None
1	0	0	None
0	1	1	$Cov(\beta_i, \tau_i), Coskew(\beta_i, \tau_i, \tau_i)$
1	0	1	$Var(\tau_i)$, $Cov(\beta_i, \tau_i)$
1	No term	1	$Cov(\beta_i, \tau_i)$
1	1	0	None
1	1	1	$Cov(\beta_i, \tau_i)$, $Var(\tau_i)$, $Coskew(\beta_i, \tau_i, \tau_i)$, $Cov(\gamma_i, \tau_i)$

Representative agent errors are fairly intuitive, and can be broken up into three broad ideas. First, if there are quadratic response terms (assumed here to be negative), even if they're homogeneous, variation in taxes will cause a representative agent model to understate the true response by an expression directly related to $Var(\tau_i)$, because the mean sum of squares will be higher than the square of the mean by Jensen's inequality. This makes sense: having two identical people taxed at 50% will have a potentially quite different aggregate response as taxing one individual at 0% and the other at 100%, but this requires nonlinearity in response.

Second, if linear responses are heterogeneous and taxes are heterogeneous, then the representative agent will under- or over-state the aggregate response depending on the sign and magnitude of $Cov(\beta_i, \tau_i)$, the covariance between responses and tax rates. This too is intuitive: if we tax more responsive people more and less responsive people less, the representative agent will understate aggregate responsiveness by shifting taxes to the less responsive and away from the more responsive. The opposite holds true if we tax the less responsive more.

Finally, if nonlinear responses are heterogeneous and taxes are heterogeneous, then the core response will not only depend on the variances of τ_i and β_i and their covariance, but also their co-skewness, the mixed third moment, due to the interaction of β_i with the squared τ_i term.

3 Three Illustrative Models

3.1 A Simple Model of Labor Supply: Prescott 2004

Take the representative agent of Prescott (2004), used to successfully predict the cross-country effect of taxes on labor supply. In a static version of the model, households have log utility over both consumption and leisure.

$$u(c_i, \ell_i) = \log(c_t) + \psi_i \log(100 - L_i)$$

Subject to the household's period budget constraint:

$$(1 + \tau_i^c)c_i = (1 - \tau_i^L)w_i L_i + (1 - \tau_i^\nu)\nu_i$$

Taking first order conditions, we can solve for L^* in closed form:

$$L_i^* = \frac{100w_i(1 - \tau_i^L) - (1 - \tau_i^\nu)\nu_i\psi_i}{w_i(1 - \tau_i^L)(1 + \psi_i)}$$

Given L_i^*, we can solve for $\beta_i = \frac{\partial L_i^*}{\partial \tau_i^L}$ and $\gamma_i = 2\frac{\partial^2 L_i^*}{\partial(\tau_i^L)^2}$ in closed form:

Taking L^*, we can solve for β_i and γ_i in closed form:

$$\beta_i = -\frac{\nu_i(1 - \tau_i^\nu)\psi_i}{w(1 - \tau_i^L)(1 + \psi_i)}$$

$$\gamma_i = -\frac{4\nu_i(1 - \tau_i^\nu)\psi_i}{w(1 - \tau_i^L)^3(1 + \psi_i)}$$

The values for ν_i, τ_i^ν, τ_i^L, and w_i are measurable, and there is a single free parameter to calibrate, ψ_i. There are three obvious choices of how to match the representative agent's ψ_R: it can match either:

- The mean labor supply (the usual choice): $\frac{1}{N}\sum_{i=1}^N L_i$
- The mean linear responsiveness to a tax change: $\frac{1}{N}\sum_{i=1}^N \beta_i$

- The mean quadratic responsiveness to a tax change: $\frac{1}{N}\sum_{i=1}^{N}\gamma_i$

Because of our simple choice of representative agent preferences, we can only match one. Other choices will be able to match all three. By matching to the mean linear responsiveness to a tax change, but not the quadratic responsiveness, we add an additional term to our error, which is the difference between the true quadratic prediction and the mis-calibrated quadratic prediction,

$$\text{Calibration error} = (\gamma_R^{\text{true}} - \gamma_R^{\text{miscalibrated}})(\Delta\tau_i)^2$$

This term is potentially quite significant, as I will show.

3.1.1 Calibration

Several parameters need to be calibrated for each household. To inform my calibration of wages, disutility of labor, and property income, I use the Current Population Survey (CPS). I extract all individuals in 2016 to directly measure the distributions for w_i, ν_i, and L_i. I estimate each marginal density as:

$$L_i \sim Mixture\left[\{0.31, 0.50, 0.16, 0.03\}, \{\mathcal{N}(23.05, 13.54), \mathcal{C}(39.99, 0.03), \ln\mathcal{N}(3.90, 0.07), \mathcal{C}(63.96, 11.78)\}\right]$$

$$\nu_i \sim Mixture\left[\{0.51, 0.49\}, \{\mathcal{C}(0.02, 0.06), -(0.38, 319)\}\right]$$

$$w_i \sim \ln\mathcal{N}(3.02, 0.67)$$

Where I do not use L_i directly in the model: I use it to calibrate the distribution of ψ_i, given other distributions. There are three other parameters to be chosen, which unfortunately are harder to estimate: τ_i, $\Delta\tau_i$, the property income tax τ_i^c, and the correlations between all variables. I assume that the marginal distributions for τ_i and $\Delta\tau_i$ are:

$$\tau_i \sim \mathcal{N}(0.175, 0.04)$$

$$\Delta\tau_i \sim \mathcal{N}(0.05, 0.03)$$

I further assume $\tau_i^c = 0.2$ for all households.

I use a normal distribution copula to join the four primal distributions:

$$\begin{bmatrix} \Phi(F^{-1}(\nu_i)) \\ \Phi(F^{-1}(\Delta\tau_i)) \\ \Phi(F^{-1}(w_i)) \\ \Phi(F^{-1}(\tau_i)) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.4 & 0 & 0 \\ 0.4 & 1 & 0.2 & 0 \\ 0 & 0.2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

Figure 1 depicts the joint distribution of β_i and γ_i. It helps make clear that while there is only a single parameter (ϕ_i) being used to (potentially) calibrate responsiveness, other variables (ν_i, in particular) play a role in determining how β_i and γ_i are separated. In other words, given w_i and τ_i, typically taken from the data, the representative agent's relationship between β_i and γ_i is constrained

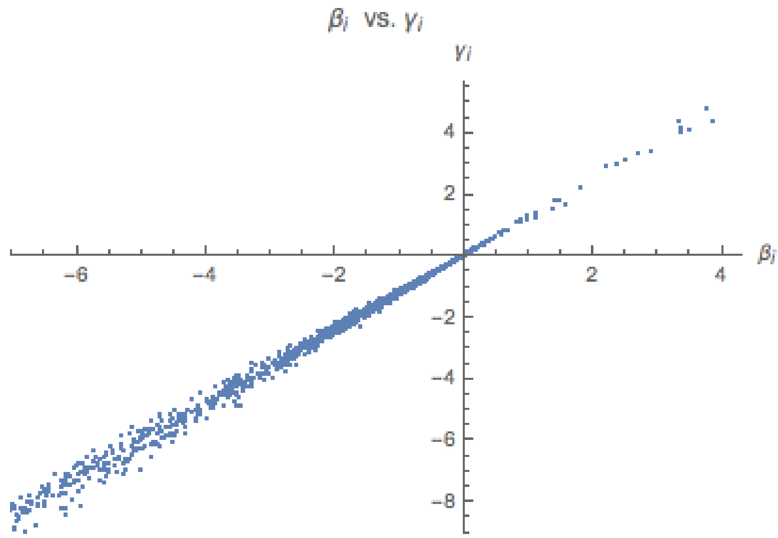

Figure 1: This depicts the distribution of both linear and quadratic responsiveness.

3.1.2 Results

Figure 4 depicts both heterogeneous-agent and representative agent responses and their approximations. The quadratic approximations do a good job of representing both models: the heterogeneous agent approximation understates the heterogeneous response by 1%, and the representative agent approximation understates the representative response by 0.4%. However, the representative agent approximation dramatically misstates the heterogeneous agent approximation: a change in tax rates of 5% with a standard deviation of 3% yields a reduction in labor of

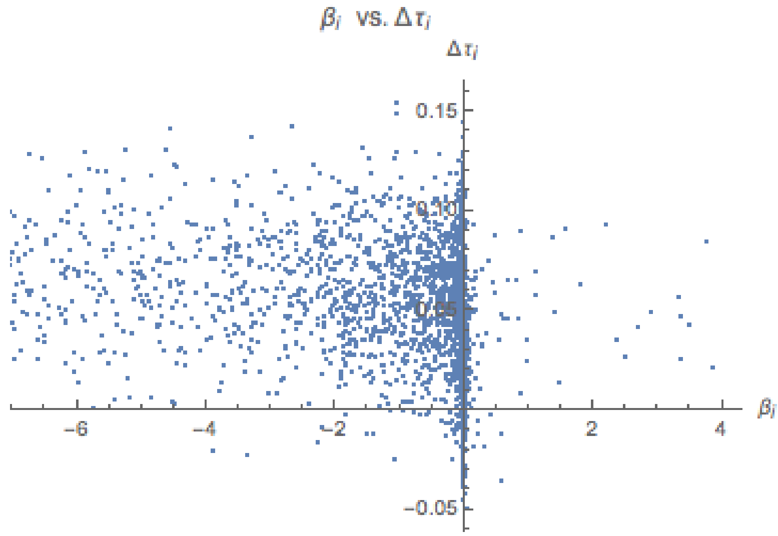

Figure 2: This depicts the distribution of both linear responsiveness and tax change.

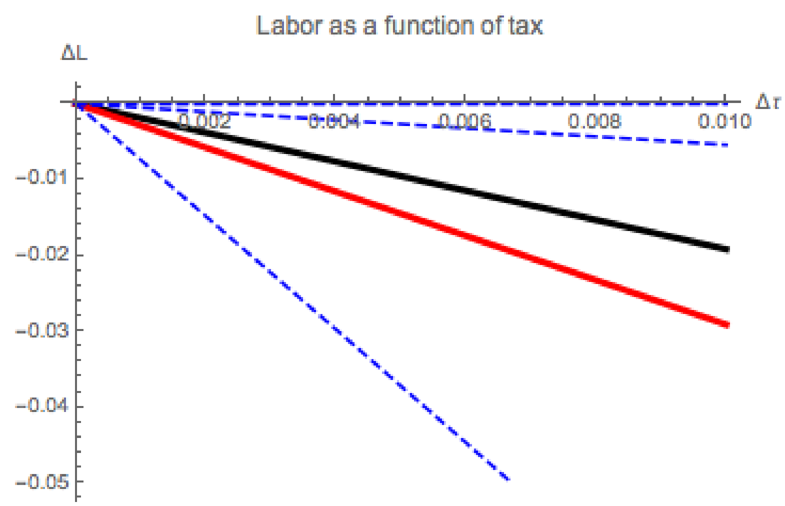

Figure 3: This depicts the distribution of labor responses as a function of tax changes.

0.21 hours in a heterogeneous agent model, but only 0.1 in a representative agent model.

Figure 6 sheds light on the sources of the approximation's failure by depicting the two approximations as well as the four sources of misstatement identified in equation 3, the two possible miscalibrations identified in equation 4, and the higher-order terms, calculated by taking the difference between the actual responses and approximations. Two major sources of misstatement are initial linear mis-calibration and the covariance between linear responsiveness and tax change: the first contributing 46% of the misstatement and the second contributing 42%. The second misstatement, which alone would increase the representative agent's response by 50%,

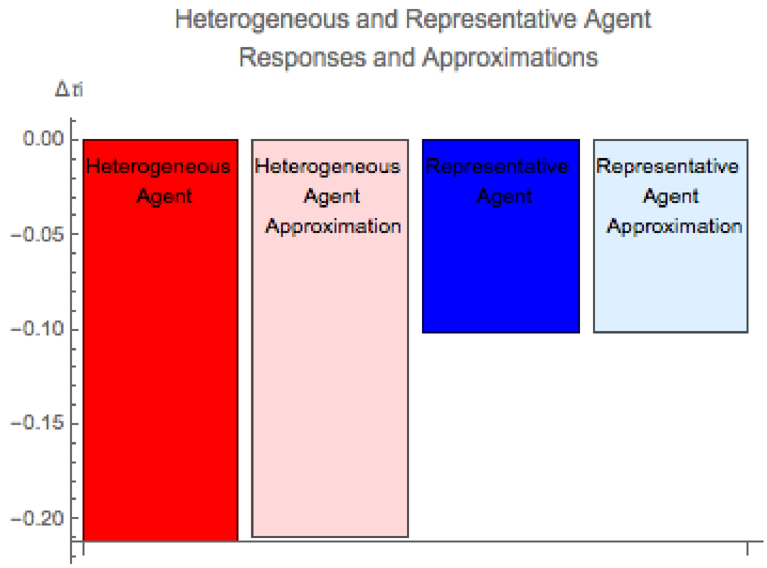

Figure 4: This depicts the heterogenous agent and representative agent responses, as well as their approximations.

is barely visible when depicted in Figure 2. Three other sources of misstatement are not insignificant: quadratic miscalibration, the covariance between the quadratic term and taxes, and the variance of taxes, increase the representative agent's response by nearly 3%, 5%, and 3% respectively.

3.2 The Basic Model of General-Equilibrium Heterogeneous-Agent Savings: Krusell and Smith (1998)

Understanding the "approximate aggregation" result of Krusell and Smith (1998) in the context of equation 3 is simple, and is reflected in their description of the savings rule, which is essentially linear in income above a certain level, as depicted in Figure 6, in which I replicate their basic findings (with a sightly different calibration). Because the slopes of capital as a function of initial assets of both types of agent are nearly identical, there is no room for β_i to play a role. While there is a small amount of nonlinearity near zero assets, this is

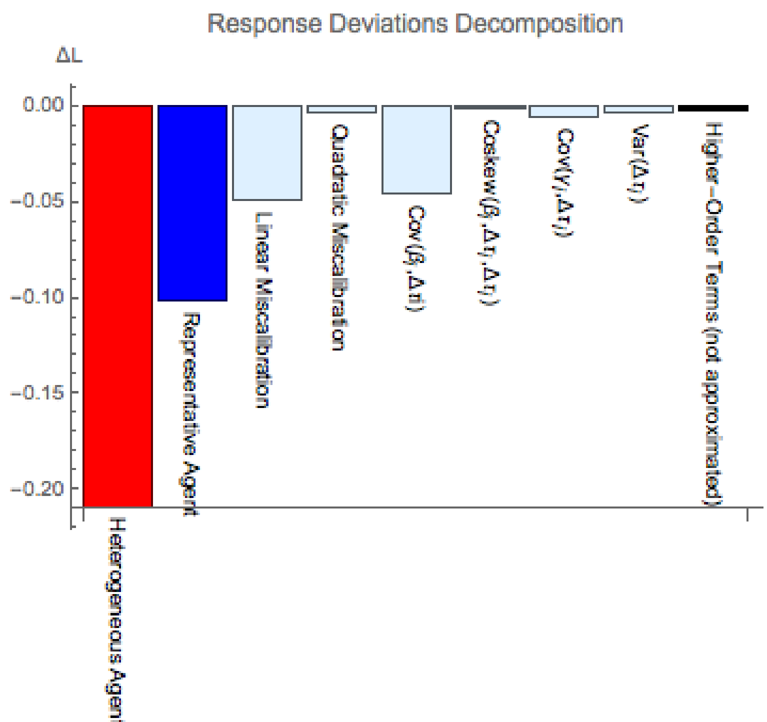

Figure 5: This decomposes the difference between the heterogeneous-agent approximation and the representative agent approximation.

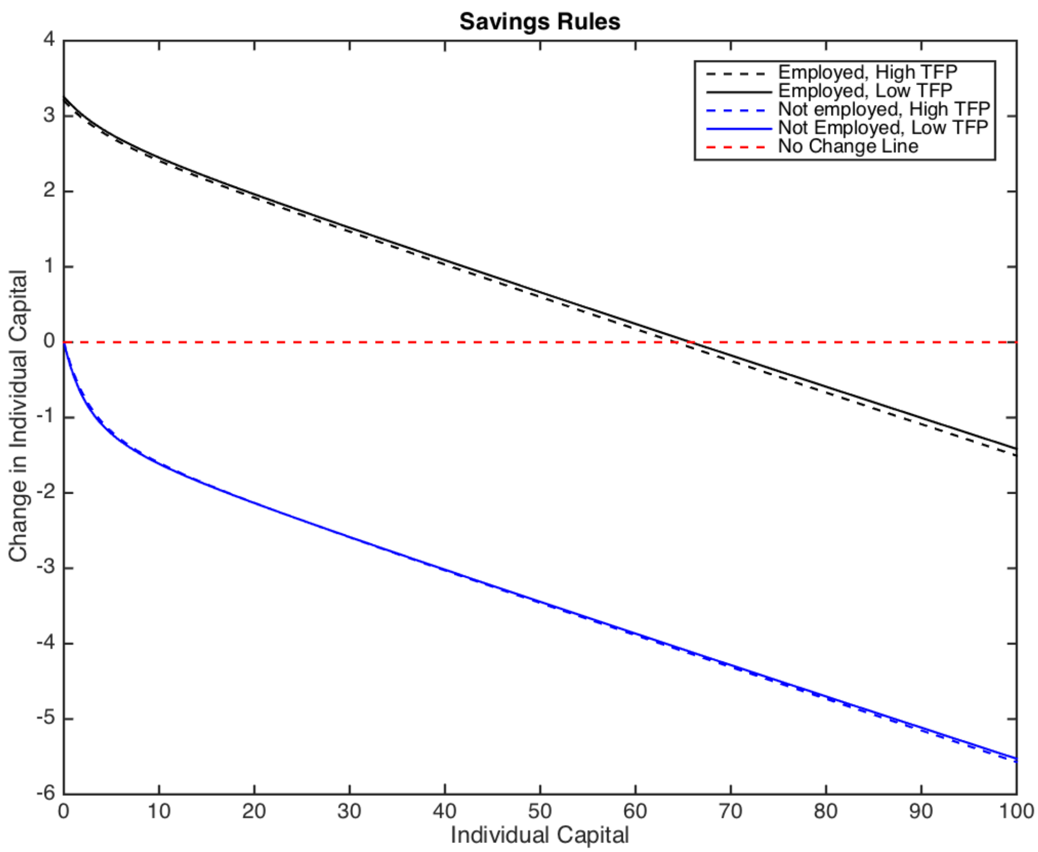
Savings Rules
Employed, High TFP
Employed, Low TFP
Not employed, High TFP
Not Employed, Low TFP
No Change Line
Change in Individual Capital
Individual Capital
4
3
2
1
0
-1
-2
-3
-4
-5
-6
0
10
20
30
40
50
60
70
80
90
100

Figure 6:

3.3 Wealthy Hand to Mouth

$$
\begin{aligned}
v_0 &= \max_{m_1, a} \log(c_1) + \log(c_2) \\
a + m_1 &= \omega \\
c_1 + m_2 &= y_1 + m_1 \\
c_2 &= y_2 + m_2 + Ra \\
m_1 &\geq 0, a \geq 0
\end{aligned}
$$

$$
\begin{aligned}
m_2 &= \max\left\{\frac{y_1 + \omega - y_2 - (1+R)a}{2}, 0\right\} \\
a &= \max\left\{\frac{R^\sigma(y_1 + \omega) - y_2}{R + R^\sigma}, 0\right\}
\end{aligned}
$$

Which gives slopes of c_1 with respect to an unexpected transfer T in the first period of:

$$
\frac{\partial c_1}{\partial T} = \begin{cases} \frac{1}{2} & \text{if } T + y_1 + y_2 + \omega > 0 \text{ and } \frac{R^\sigma y_1 - y_2 + R^\sigma \omega}{R + R^\sigma} \\ -\frac{R + R^\sigma}{2(R + R^\sigma)} & \text{o.w.} \end{cases}
$$

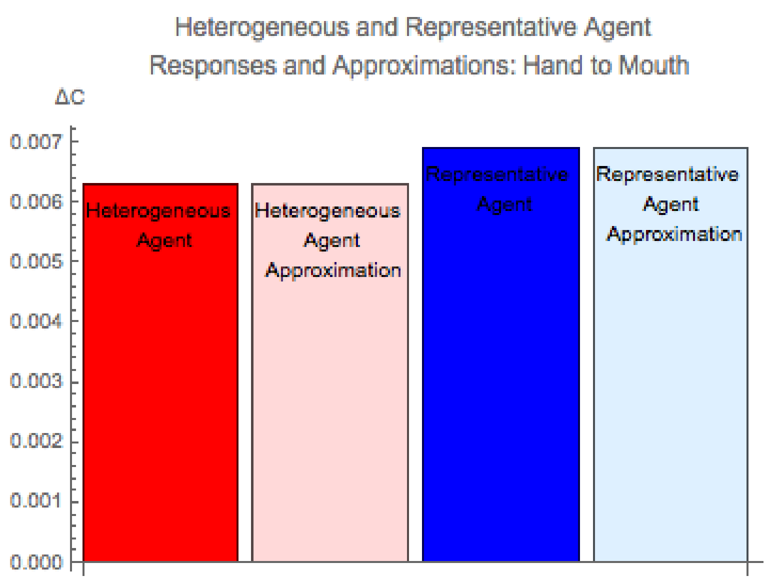

Figure 7: This depicts the heterogenous agent and representative agent responses, as well as their approximations.

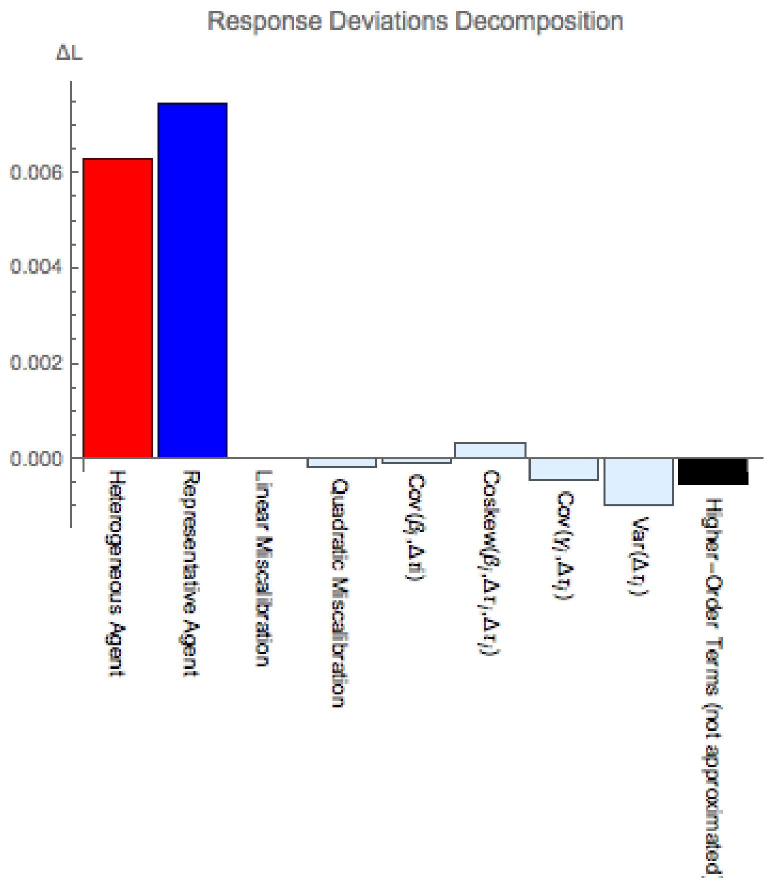

Figure 8: This decomposes the difference between the heterogeneous-agent approximation and the representative agent approximation.

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