Gabriel’s Horn

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February 2016

We shall observe the mathematical phenomenon Gabriel’s horn (Also known as Toricelli’s trumpet) and investigate its seemingly paradoxical properties. The name refers to the Archangel Gabriel blowing his horn to announce the Day of Judgement. The name is fitting as it serves as a bridge between the divinity and mortality in a biblical sense and as the bridge between infiniteness and finiteness mathematically. The shape was discovered and first studied by Italian mathematician Evangelista Toricelli (Hence the second name) in the early 1600’s. The geometric figure lies in 3-dimensional space and we shall construct it below.

Consider the graph of the function $f(x) = \frac{1}{x}$ pictured below:
Consider the solid obtained by rotating the function from \( x=1 \) onward about the \( x \)-axis and the result we obtain is shown below.

Notice how the image above looks like an infinitely long herald trumpet (Hence the name). We now begin to investigate some of the properties of such a shape.

First we consider the volume of this figure. We know that because the original function was rotated about the \( x \)-axis, the volume is given by:

\[
V = \pi \int_{1}^{\infty} \left( \frac{1}{x} \right)^2 \, dx = \lim_{b \to \infty} \pi \int_{1}^{b} \left( \frac{1}{x} \right)^2 \, dx
\]

\[
= \lim_{b \to \infty} \pi \int_{1}^{b} \frac{1}{x^2} \, dx
\]

\[
= \lim_{b \to \infty} \pi \left( -\frac{1}{x} \right) \bigg|_{1}^{b}
\]

\[
= \lim_{b \to \infty} \pi \left( \left( -\frac{1}{b} \right) - (-1) \right)
\]

\[
= \pi
\]

Surprisingly, our horn actually has finite volume despite being infinitely long. This does not alarm us as the impossible is just another day in the life of a mathematician. We now consider the surface area of our trumpet.
From the arc length formula and rotation about the x-axis, we can evaluate the surface area of the figure by the following improper integral:

\[
SA = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \left( \frac{d}{dx} \left[ \frac{1}{x} \right] \right)^2} \, dx = \lim_{b \to \infty} 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \left( \frac{-1}{x^2} \right)^2} \, dx
\]

\[
= \lim_{b \to \infty} 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx
\]

We now run into a problem with the above integral. Perhaps one could make an appropriate trigonometric substitution (and several more after...) or use a method of approximate integration, but for simplicity’s sake and all that we’re concerned with, we will apply a simple comparison.

Because for all \( x \geq 1 \), we see that \( \frac{1}{x^4} \geq 0 \), and so it intuitively follows that for \( x \geq 1 \), \( \sqrt{1 + \frac{1}{x^4}} \geq \sqrt{1 + 0} = 1 \) and so it follows that:

\[
SA = \lim_{b \to \infty} 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx \geq \lim_{b \to \infty} 2\pi \int_1^b \frac{1}{x} \, dx
\]

\[
= \lim_{b \to \infty} 2\pi \ln |x|_1^b
\]

\[
= \lim_{b \to \infty} 2\pi \ln(b)
\]

\[
= \infty
\]

Therefore, by comparison, the surface area of Gabriel’s Horn is infinite. Thus we have constructed a shape with infinite surface area and yet finite volume. This is not what we would expect and surely this bizarre shape is one-of-a-kind. For instance, with these conclusions, we see it is possible to completely fill our horn with paint and yet we are unable to paint the outside of it.