Look for the Similarities
Up until this point, we have a lot of very useful tools we can use to test series. Sadly we are also very limited with these tools. What are we to do if we run across something that is not a p-series, a geometric series, or something we can not integrate?

We don’t have a specific test for every type of series, but if we find ourselves saying: “It kinda looks like this or that” then maybe we can do something. This brings us to the idea of comparing series to try to find similar behavior.

The Direct Comparison Test
Our first though is to compare series by size. Consider two series of all positive terms where $a_n \geq 0$ and $b_n \geq 0$. Suppose we have the relation:

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$$

If we have a series of all positive terms, it makes logical sense that as we add more and more terms, the series will get larger and larger.

Suppose the righthand series converges. This means that the lefthand series has an upper bound. As we add more and more terms it gets bigger and bigger and yet it is still bounded above by something fixed. It then makes sense that the leftmost series must converge.

On the other hand suppose the leftmost series diverges. If a series of all positive terms diverges, it logically follows that it must become infinitely large. If the equality holds then the rightmost series is larger than something infinitely large. Therefore it must also be infintely large and therefore must diverge.

The above conclusions give us what we call the Direct Comparison Test
If \( a_n \geq 0 \) and \( b_n \geq 0 \) for all \( n \) and:

\[
\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n
\]

If \( \sum_{n=1}^{\infty} b_n \) converges \( \Rightarrow \sum_{n=1}^{\infty} a_n \) converges

If \( \sum_{n=1}^{\infty} a_n \) diverges \( \Rightarrow \sum_{n=1}^{\infty} b_n \) diverges

The above is a very useful tool, but it can have its drawbacks. For instance we might try to compare two series but not have the correct inequality. Clearly being less than something infinite or being greater than something finite does not mean much. For instance:

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n}
\]

And yet we know from the \textbf{P-Test} that the leftmost series converges and the rightmost series diverges. If we run into a mismatched inequality, we must either try to apply another test or try to directly compare to something else.

for instances like the above, we introduce a stronger comparison test.

\textbf{The Limit Comparison Test}

The idea behind this test is that maybe as we add more and more terms and as \( n \) gets large two series will look more and more alike. The proof behind this test is rather rigorous so I’ll forgo it.

Given the following series such that \( a_n \geq 0 \) and \( b_n \geq 0 \) for all \( n \).

\[
\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n
\]

If we can show:
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = c
\]

Where \(0 < c\) and \(c\) is a finite number, then we can infer the two series will behave similarly. If, for instance, \(c = 0\) or \(c\) is not finite, we can infer nothing and must either apply another test or try to compare our original series to something else.