Chapter 7: The Alternating Series Test

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Nonnegative Nonsense
Up until this point we have only observed series of all positive terms and now we turn our attention to those with a regular mix of positive and negative terms. In some ways alternating series are simpler and easier to handle as there is a more straight-forward way to proceed.

We start off with the statement of the Alternating Series Test. The proof is fairly rigorous so we will ignore it. Given an alternating series of the form:

\[ \sum_{n=1}^{\infty} (-1)^n b_n \]

If \( b_n \) is a decreasing sequence and \( \lim_{n \to \infty} b_n = 0 \)

Then the alternating series will converge.

As a side note: If either of the above criteria fail, the series must be divergent. For instance we already know if the limit is nonzero the series must diverge by the Test for Divergence. Secondly, we really need the terms to decrease monotonically after a certain point. It’s fine if the terms increase for the first few \( n \)-values and then ultimately decrease. Adding and subtracting a few finite terms won’t change the outcome.

Overall this test seems pretty straightforward as we only need to check for two things. Clearly the second part is the easiest of the two as we are already familiar with many methods of taking limits. The first part can be a little tricky, but there are two methods that we can apply to show a sequence is decreasing.

The first is relatively simple. Given a sequence \( b_n \), If we can show that for all \( n \) (after a certain point) that:
\[ b_{n+1} \leq b_n \]

Recursively we see that each term is smaller than the previous, so our sequence must be monotonically decreasing. For instance if we have some kind of sequence of rational terms with a fixed numerator, if we can show that denominator increases then the sequence must decrease.

Alternatively, if it is difficult to determine the above relation between the \( n^{th} \) and \( (n+1)^{st} \) term and our sequence is given by a continuous function we use its first derivative.

Given a sequence \( b_n = f(n) \) if \( f \) is a continuous function and \( \frac{df}{dx} < 0 \) after a certain point \( k \) or equivalently on the interval \([k, \infty)\) then the sequence will ultimately be decreasing.

Just as a recap since it’s been awhile since we’ve taken a derivative:

\[
\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)
\]

\[
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\]