Converge? Oh, Absolutely
Now that we have explored a lot of different types of series and are very familiar with convergence we ask ask ourselves: Are there different types of convergence? Are there any stronger implications?

We introduce the notion of Absolute Convergence.

If \( \sum_{n=1}^{\infty} a_n \) converges and \( \sum_{n=1}^{\infty} |a_n| \) converges, we say \( \sum_{n=1}^{\infty} a_n \) converges absolutely.

Clearly if we have a series of all non-negative terms, convergence and absolute convergence are equivalent. However, it is a different story if we deal with series of infinitely many negative terms, particularly alternating series.

We introduce the notion of Conditional Convergence

If \( \sum_{n=1}^{\infty} a_n \) converges but \( \sum_{n=1}^{\infty} |a_n| \) diverges, we say \( \sum_{n=1}^{\infty} a_n \) converges conditionally.

We reserve conditional convergence for alternating series only. As mentioned before the only type of convergence for series of all positive terms is absolute.

So our current so far is: If we have a non-alternating series, we test for convergence as usual, but if we have an alternating series we have to do a little more work.

Given an alternating series of the form:
If \( \sum_{n=1}^{\infty} (-1)^n b_n \) where \( b_n \geq 0 \) for all \( n \)

We apply the Alternating Series Test as normal. If our series fails and the series diverges, we are finished, but if our series does pass the Alternating Series Test we know our series is some type of convergent and must now consider the convergence of

\[
\sum_{n=1}^{\infty} |(-1)^n b_n| = \sum_{n=1}^{\infty} b_n \quad \text{as} \quad b_n \geq 0
\]

Now that we are dealing with a series of all positive terms, we can use any of our previous tests (except for the Alternating Series Test of course) to investigate the convergence of the above series.

For instance, the best example of a conditionally convergent series is the alternating harmonic series.

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}
\]

Clearly \( \lim_{n \to \infty} \frac{1}{n} = 0 \) and since \( \frac{1}{n+1} < \frac{1}{n} \) the sequence is decreasing.

Therefore, by the Alternating Series Test, the series converges and because:

\[
\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{which diverges}
\]

The alternating harmonic series is conditionally convergent.