Chapter 9: The Ratio and Root Test

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The Ratio Test
Under the notion of absolute convergence we ask ourselves: "Does there exist a test that tells determines absolute convergence on its own for any series (even alternating)?" The answer is yes, there does exist such a test and in fact there exist two. The first is called the Ratio Test. Given a series:

\[ \sum_{n=1}^{\infty} a_n \] we look at \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \)

if \( L < 1 \) the series converges absolutely. If \( L > 1 \) the series diverges. If \( L = 1 \) the test is inconclusive, we can infer nothing, and must apply a different test.

This is a very powerful and versatile test, however we must fully understand the implications. The Ratio Test tells us only if a series diverges or converges absolutely. The limit being equal to one does not imply conditional convergence (for instance try any convergent or divergent p-series) and only tells us we need to try another test.

This test is particularly useful when dealing with series of factorials, powers of \( n \), and any mix of the two.

The Root Test
As previously mentioned, the Ratio Test is very useful, but like all the test before it, it has it does have its drawbacks. One particular difficulty is when dealing with \( n^{th} \) powers of expressions of \( n \) itself. For instance, testing the series:

\[ \sum_{n=1}^{\infty} \frac{(2n)^n(n+1)^n}{(n!)^n} \]
Would prove to be very tedious and could be computationally rigorous. For times like these, we use a test that allows us to bypass all of those annoying powers of $n$. We introduce what is known as the **Root Test**. Given a series:

$$
\sum_{n=1}^{\infty} a_n \quad \text{we look at} \quad \lim_{n \to \infty} \sqrt[n]{|a_n|} = L
$$

if $L < 1$ the series converges absolutely. If $L > 1$ the series diverges. If $L = 1$ the test is inconclusive, we can infer nothing, and must apply a different test.

As mentioned before, this can a be a particularly useful test when working with various $n^{th}$ powers. To make this test a little more versatile we do have a limit identity that may prove to be useful:

$$
\lim_{n \to \infty} \sqrt[n]{n} = 1
$$