The international Terrestrial Reference System: ITRS

- Definition adopted by the IUGG and IAG: see http://tai.bipm.org/iers/conv2003/conv2003.html
- Tri-dimensional orthogonal (X,Y,Z), equatorial (Z-axis coincides with Earth’s rotation axis)
- Non-rotating (actually, rotates with the Earth)
- Geocentric: origin = Earth’s center of mass, including oceans and atmosphere.
- Units = meter and second S.I.
- Orientation given by BIH at 1984.0.
- Time evolution of the orientation ensured by imposing a no-net-rotation condition for horizontal motions.
The no-net-rotation (NNR) condition

- **Objective:**
  - Representing velocities without referring to a particular plate.
  - Solve a datum defect problem: ex. of 2 plates \( \Rightarrow 1 \) relative velocity to solve for 2 “absolute” velocities… (what about 3 plates?)

- The no-net-rotation condition states that the total angular momentum of all tectonic plates should be zero.

- See figure for the simple (and theoretical) case of 2 plates on a circle.

- The NNR condition has no impact on relative plate velocities.

- It is an additional condition used to define a reference for plate motions that is not attached to any particular plate.

\[
M_A = R \times V_{A/NNR} \quad M_B = R \times V_{A/NNR} \\
\Sigma M = 0 \quad \Rightarrow V_{B/NNR} = -V_{A/NNR} = V_{B/A} / 2
\]
The Tisserand reference system

- “Mean” coordinate system in which deformations of the Earth do not contribute to the global kinetic moment (important in Earth rotation theory)

- Let us assume two systems $R$ (inertial) and $R_o$ (translates and rotates w.r.t. $R$). Body $E$ is attached to $R_o$. At point $M$, one can write:

\[
\begin{align*}
\vec{R} &= \vec{R}_o + \vec{r} \\
\vec{V} &= \vec{V}_o + \vec{v} + \vec{\omega} \times \vec{r}
\end{align*}
\]

- One can show that the Tisserand condition is equivalent to:

\[
\begin{align*}
\int_E \vec{v} \ dm &= \vec{0} & \text{No translation condition} \\
\int_E \vec{v} \times \vec{r} \ dm &= \vec{0} & \text{No rotation condition}
\end{align*}
\]
The Tisserand reference system

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\end{align*}
\]

• The system of axis defined by the above conditions is called “Tisserand system”.

• Integration domain:
  – Should be entire Earth volume
  – But velocities at surface only => integration over surface only

• With hypothesis of spherical Earth + uniform density, volume integral becomes a surface integral
The NNR reference system

- The Tisserand no-rotation conditions is also called “no-net-rotation” condition (NNR)

- For a spherical Earth of unit radius and uniform density, the NNR conditions writes:

\[ \int_S \vec{r} \times \vec{v} \, dA = 0 \]

- The integral can be broken into a sum to account for discrete plates:

\[ \int_S \vec{r} \times \vec{v} \, dA = \sum_P \int_P \vec{r} \times \vec{v} \, dA \]

- With, for a given plate:

\[ L_P = \int_P \vec{r} \times \vec{v} \, dA \]
The NNR reference system

- Assuming rigid plates, velocity at point \( M \) (position vector \( r \) in NNR) on plate \( P \) is given by:

\[
\vec{v}(\vec{r}) = \vec{\omega}_P \times \vec{r} \quad \Rightarrow \quad L_P = \int \vec{r} \times (\vec{\omega}_P \times \vec{r}) \, dA
\]

- Developing the vector product with the triple product expansion gives:

\[
L_P = \int ((\vec{r} \cdot \vec{r})\vec{\omega}_P - (\vec{r} \cdot \vec{\omega}_P)\vec{r}) \, dA = \int (\vec{r} \cdot \vec{r})\vec{\omega}_P \, dA - \int (\vec{r} \cdot \vec{\omega}_P)\vec{r} \, dA
\]

- Assuming a spherical Earth of unit radius \( \rho = 1 \), the first term introduces the plate area \( A_P \):

\[
\int (\vec{r} \cdot \vec{r})\vec{\omega}_P \, dA = \rho^2 \vec{\omega}_P \int dA = \vec{\omega}_P A_P
\]

- Dealing with the second term is a bit more involved, see next.
The NNR reference system

\[(\vec{r} \, \vec{\omega}_p) \vec{r} = (x_1 \omega_1 + x_2 \omega_2 + x_3 \omega_3) \vec{r}\]

\[= \begin{bmatrix} x_1^2 \omega_1 + x_1 x_2 \omega_2 + x_1 x_3 \omega_3 \\ x_1 x_2 \omega_1 + x_2^2 \omega_2 + x_2 x_3 \omega_3 \\ x_1 x_3 \omega_1 + x_2 x_3 \omega_2 + x_3^2 \omega_3 \end{bmatrix} \]

Therefore:

\[
\int_P (\vec{r} \, \vec{\omega}_p) \vec{r} \, dA = \begin{bmatrix} \int x_1^2 & \int x_1 x_2 & \int x_1 x_3 \\ \int x_1 x_2 & \int x_2^2 & \int x_2 x_3 \\ \int x_1 x_3 & \int x_2 x_3 & \int x_3^2 \end{bmatrix} \vec{\omega}_p \, dA
\]

We introduce a 3x3 symmetric matrix \(S_p\) with elements defined by:

\[S_{ij} = \int_P (x_i \, x_j) \, dA\]

Therefore the integral becomes:

\[
\int_P (\vec{r} \, \vec{\omega}_p) \vec{r} \, dA = S_p \, \vec{\omega}_p
\]
The NNR reference system

• Finally: \[ L_p = \int_P (\vec{r} \cdot \vec{r}) \vec{\omega}_p \, dA - \int_P (\vec{r} \cdot \vec{\omega}_p) \vec{r} \, dA \]

• Reduces to: \[ L_p = \vec{\omega}_p A_p - S_p \vec{\omega}_p \]
  \[= (A_p I - S_p) \vec{\omega}_p \]
  \[= Q_p \vec{\omega}_p \]

• With: \[ Q_p = A_p I - S_p \]

• \( Q_p \) is a 3x3 matrix that only depends on the plate geometry, with its components defined by:

\[ Q_{Pij} = \int_P \left( \delta_{ij} - x_i x_j \right) dA \]

Kronecker delta: \[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]
The NNR reference system

- The non-rotation condition:
  \[ \int_S \hat{r} \times \vec{v} \, dA = \sum_p \int_{A_p} \hat{r} \times \vec{v} \, dA = \bar{0} \]
- Becomes:
  \[ \sum_p Q_p \bar{\omega}_p = \bar{0} \]
- Now, observations are relative plate motions, for instance plate \( P \) w.r.t. Pacific plate. Angular velocities are additive, one can then write:
  \[ \bar{\omega}_{P/\text{NNR}} = \bar{\omega}_{P/\text{Pacific}} + \bar{\omega}_{\text{Pacific/NNR}} \]
- Therefore:
  \[ \sum_p Q_p \left( \bar{\omega}_{P/\text{Pacific}} + \bar{\omega}_{\text{Pacific/NNR}} \right) = \bar{0} \]
  \[ \Rightarrow \sum_p Q_p \bar{\omega}_{P/\text{Pacific}} + \sum_p Q_p \bar{\omega}_{\text{Pacific/NNR}} = \bar{0} \]
  \[ \Rightarrow \sum_p Q_p \bar{\omega}_{P/\text{Pacific}} + \frac{8\pi}{3} I \bar{\omega}_{\text{Pacific/NNR}} = \bar{0} \]

(because on a unit radius sphere: \( \sum_p Q_p = \frac{8\pi}{3} I \))
The NNR reference system

• Finally, the angular velocity of the Pacific plate w.r.t. NNR can be calculated using:

\[
\vec{\omega}_{\text{Pacific}/\text{NNR}} = -\frac{3}{8\pi} \sum P Q_P \vec{\omega}_{P/\text{Pacific}}
\]

(\(\omega_{P/\text{Pacific}}\) are known from a relative plate model, \(Q_P\) are computed for each plate from its geometry)

• Once the angular velocity of the Pacific plate in NNR is found, the angular velocity of any plate \(P\) can be computed using:

\[
\vec{\omega}_{P/\text{NNR}} = \vec{\omega}_{P/\text{Pacific}} + \vec{\omega}_{\text{Pacific}/\text{NNR}}
\]

• This method is the one used to compute the NNR-NUVEL1A model (Argus and Gordon, 1991).
Summary

- Geodetic observations face datum defect problem => need for a reference frame.

- Reference frame in modern space geodesy best implemented using minimal constraints after combination with global solutions (unless regional solution sought).

- Once global position/velocity solution is obtained, question remains of how to express it in a frame independent from any plate = no-net-rotation frame, derived from Tisserand reference system.
The no-net-rotation (NNR) condition

• The NNR condition actually has a “dynamic” origin.
• First proposed by Lliboutry (1977) as an approximation of a reference frame where moment of forces acting on lower mantle is zero.
• In its original definition, this implies:
  – Rigid lower mantle
  – Uniform thickness lithosphere
  – No lateral viscosity variations in upper mantle
  ⇒ NNR is a frame in which the internal dynamics of the mantle is null.
• These conditions are not realistic geophysically, in particular because of slabs in upper and lower mantle, that contribute greatly to driving plate motions (Lithgow-Bertelloni and Richards, 1995)
• But that’s ok, as long as NNR is simply used as a conventional reference.
The international Terrestrial Reference Frame: ITRF

- Positions (at a given epoch) and velocities of a set of geodetic sites (+ associated covariance information) = \textit{dynamic} datum

- Positions and velocities estimated by combining independent geodetic solutions and techniques.

- Combination:
  - “Randomizes” systematic errors associated with each individual solutions
  - Provides a way of detecting blunders in individual solutions
  - Accuracy is equally important as precision

- 1984: VLBI, SLR, LLR, Transit
- 1988: TRF activity becomes part of the IERS => first ITRF = ITRF88
- Since then: ITRF89, ITRF90, ITRF92, ITRF93, ITRF94, ITRF96, ITRF97, ITRF2000
- \textbf{Current = ITRF2005}:
  - Up to 25 years of data
  - GPS sites defining the ITRF are all IGS sites
  - Wrms on velocities in the combination: 1 mm/yr VLBI, 1-3 mm/yr SLR and GPS
  - Solutions used: 3 VLBI, 1 LLR, 7 SLR, 6 GPS, 2 DORIS

- ITRF improves as:
  - Number of sites with long time series increases
  - New techniques appear
  - Estimation procedures are improved
The international Terrestrial Reference Frame: ITRF

- Apply minimum constraints equally to all loosely constrained solutions: this is the case of SLR and DORIS solutions.
- Apply No-Net-Translation and No-Net-Rotation condition to IVS solutions provided under the form of Normal Equation.
- Use as they are minimally constrained solutions: this is the case of IGS weekly solutions.
- Form per-technique combinations (TRF + EOP), by rigorously staking the time series, solving for station positions, velocities, EOPs and 7 transformation parameters for each weekly (daily in case of VLBI) solution w.r.t the per-technique cumulative solution.
- Identify and reject/de-weight outliers and properly handle discontinuities using piece-wise approach.
- Combine if necessary cumulative solutions of a given technique into a unique solution: this is the case of the two DORIS solutions.
- Combine the per-technique combinations adding local ties in co-location sites.
The international Terrestrial Reference Frame: ITRF

- **Origin**: The ITRF2005 origin is defined in such a way that there are null translation parameters at epoch 2000.0 and null translation rates between the ITRF2005 and the ILRS SLR time series.

- **Scale**: The ITRF2005 scale is defined in such a way that there are null scale factor at epoch 2000.0 and null scale rate between the ITRF2005 and IVS VLBI time series.

- **Orientation**: The ITRF2005 orientation is defined in such a way that there are null rotation parameters at epoch 2000.0 and null rotation rates between the ITRF2005 and ITRF2000. These two conditions are applied over a core network.
ITRF in practice

• Multi-technique combination.
• Origin = SLR, scale = VLBI, orientation = all.
• Position/velocity solution.
• Velocities expressed in no-net-rotation frame:
  – ITRF2000: minimize global rotation w.r.t. NNR-NUVEL1A using 50 high-quality sites far from plate boundaries
  – Subtlety: ITRF does not exactly fulfill a NNR condition because Nuvel1A is biased...
• Provided as tables (position, velocities, uncertainties)
• Full description provided as SINEX file (Solution Independent Exchange format): ancillary information + vector of unknowns + full variance-covariance matrix (i.e. with correlations).
ITRF in practice

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