ECE 301: Signals and Systems
Homework Solution #1

Professor: *Aly El Gamal*
TA: *Xianglun Mao*
Problem 1

Determine the values of $P_\infty$ and $E_\infty$ for each of the following signals:

(a) $x_1(t) = e^{-2t}u(t)$

(b) $x_2(t) = e^{j(2t+\pi/4)}$

(c) $x_3(t) = \cos(t)$

(d) $x_1[n] = (\frac{1}{2})^n u[n]$

(e) $x_2[n] = e^{j(\pi/2n+\pi/8)}$

(f) $x_3[n] = \cos(\frac{\pi}{4} n)$

Solution

(a) $E_\infty = \int_0^\infty e^{-2t} dt = \frac{1}{2}$. $P_\infty = 0$, because $E_\infty < \infty$.

(b) $x_2(t) = e^{j(2t+\pi/4)}$, $|x_2(t)| = 1$. Therefore,

$$E_\infty = \int_{-\infty}^{\infty} |x_2(t)|^2 dt = \int_{-\infty}^{\infty} dt = \infty.$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x_2(t)|^2 dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt = \lim_{T \to \infty} 1 = 1.$$

(c) $x_3(t) = \cos(t)$. Therefore,

$$E_\infty = \int_{-\infty}^{\infty} |x_3(t)|^2 dt = \int_{-\infty}^{\infty} \cos^2(t) dt = \infty.$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x_3(t)|^2 dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \cos^2(t) dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(1 + \cos(2t)\right) dt = \frac{1}{2}.$$

(d) $x_1[n] = (\frac{1}{2})^n u[n]$, $|x_1[n]|^2 = (\frac{1}{4})^n u[n]$. Therefore,

$$E_\infty = \sum_{n=-\infty}^{\infty} |x_1[n]|^2 = \sum_{n=0}^{\infty} (\frac{1}{4})^n = \frac{4}{3}.$$

$$P_\infty = 0$$, because $E_\infty < \infty$.

(e) $x_2[n] = e^{j(\pi/2n+\pi/8)}$, $|x_2[n]|^2 = 1$. Therefore,

$$E_\infty = \sum_{n=-\infty}^{\infty} |x_2[n]|^2 = \sum_{n=-n}^{\infty} 1 = \infty.$$

$$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_2[n]|^2 = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 = 1.$$

(f) $x_3[n] = \cos(\frac{\pi}{4} n)$. Therefore,

$$E_\infty = \sum_{n=-\infty}^{\infty} |x_3[n]|^2 = \sum_{n=-\infty}^{\infty} \cos^2(\frac{\pi}{4} n) = \infty.$$

$$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_3[n]|^2 = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2(\frac{\pi}{4} n) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(1 + \cos(\frac{\pi}{2} n)\right) = \frac{1}{2}.$$
Problem 2

A continuous-time signal $x(t)$ is shown in Figure 6. Sketch and label carefully each of the following signals:

(a) $x(4 - \frac{t}{2})$
(b) $[x(t) + x(-t)]u(t)$
(c) $x(t)[\delta(t + \frac{3}{2}) - \delta(t - \frac{3}{2})]$

Figure 1: The continuous-time signal $x(t)$.

Solution

Figure 2: Sketches for the resulting signals.
Problem 3

A discrete-time signal $x[n]$ is shown in Figure 3. Sketch and label carefully each of the following signals:

(a) $x[3n]$

(b) $x[n]u[3 - n]$

(c) $x[n - 2]δ[n - 2]$

Solution

Figure 3: The discrete-time signal $x[n]$.

Figure 4: Sketches for the resulting signals.
Problem 4

Determine and sketch the even and odd parts of the signals depicted in Figure 5. Label your sketches carefully.

Solution

Figure 5: The continuous-time signal $x(t)$.

Figure 6: Sketches for the resulting signals.
Problem 5

Let \( x(t) \) be the continuous-time complex exponential signal

\[
x(t) = e^{j\omega_0 t}
\]

with fundamental frequency \( \omega_0 \) and fundamental period \( T_0 = 2\pi/\omega_0 \). Consider the discrete-time signal obtained by taking equally spaced samples of \( x(t) \) - that is,

\[
x[n] = x(nT) = e^{j\omega_0 nT}
\]

(a) Show that \( x[n] \) is periodic if and only if \( T/T_0 \) is a rational number - that is, if and only if some multiple of the sampling interval exactly equals a multiple of the period of \( x(t) \).

(b) Suppose that \( x[n] \) is periodic - that is, that

\[
\frac{T}{T_0} = \frac{p}{q}
\]

where \( p \) and \( q \) are integers. What are the fundamental period and fundamental frequency of \( x[n] \)? Express the fundamental frequency as a fraction of \( \omega_0 T \).

(c) Again assuming that \( T/T_0 \) satisfies equation (1), determine precisely how many periods of \( x(t) \) are needed to obtain the samples that form a single period of \( x[n] \).

Solution

(a) If \( x[n] \) is periodic, then \( e^{j\omega_0 (n+N)T} = e^{j\omega_0 nT} \), where \( \omega_0 = 2\pi/T_0 \). This implies that

\[
\frac{2\pi}{T_0} NT = 2\pi k \Rightarrow \frac{T}{T_0} = \frac{k}{N} = \text{a rational number}.
\]

If \( \frac{T}{T_0} = \frac{k}{N} = \text{a rational number} \), then we have

\[
\frac{T}{T_0} = \frac{k}{N} \Rightarrow \frac{2\pi}{T_0} NT = 2\pi k.
\]

This implies that \( e^{j\omega_0 (n+N)T} = e^{j\omega_0 nT} \), where \( \omega_0 = 2\pi/T_0 \). \( x[n] \) is periodic.

Combining the above two conditions, we can conclude that \( x[n] \) is periodic if and only if \( T/T_0 \) is a rational number.

(b) If \( \frac{T}{T_0} = \frac{p}{q} \) then \( x[n] = e^{j2\pi n \frac{p}{q}} \). The fundamental period is \( N = q/gcd(p,q) \) (gcd refer to the greatest common divisor). The fundamental frequency is

\[
\frac{2\pi}{q} gcd(p,q) = \frac{2\pi p}{p q} gcd(p,q) = \frac{\omega_0 T}{p} gcd(p,q)
\]

(c) We know that the fundamental period of (b) is \( N = q/gcd(p,q) \), so overall \( \frac{NT}{T_0} = p/gcd(p,q) \) periods of \( x(t) \) is needed to obtain the samples that form a single period of \( x[n] \).