Special Distributions #1
ECON 670

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Road Map

- The Binomial and Poisson Distributions
- The Uniform and Gamma Distributions
- The Normal Distribution
- The Student’s $t$ and $F$ Distributions
The Binomial Distribution

Definition

A Bernoulli experiment is a random experiment with two mutually exclusive and exhaustive outcomes, a success or a failure.

A sequence of Bernoulli trials occurs when a Bernoulli experiment is performed several independent times so that the probability of success, say $p$, is the same from trial to trial. Let $X$ be a random variable defined as

$$X = \begin{cases} 1 & \text{with probability } p \ [\text{success}] \\ 0 & \text{with probability } 1 - p \ [\text{failure}] \end{cases}$$

Then $X \sim \text{Bernoulli}(p)$ with pmf

\[\]
Repeated tosses of a coin constitute a simple example of a sequence of Bernoulli trials.

The mean and variance of $X$ are computed as follows:

1.

2.
Many experiments can be modeled as a sequence of Bernoulli trials. If \( n \) identical Bernoulli trials are performed, define

\[ A_i = \{X = 1 \text{ on the } i\text{-th trial}\}, \quad i = 1, 2, \ldots, n \]

Assume that \( A_1, \ldots, A_n \) are independent events.

Define \( Y = \text{total } \# \text{ successes in } n \text{ trials} \). Then \( \{Y = y\} \) will occur only if, out of \( A_1, \ldots, A_n \), exactly \( y \) of them occur, and necessarily \( (n - y) \) of them do not occur.

One particular outcome may be

\[
P(A_1 \cap A_2 \cap A_3^c \cap \ldots \cap A_{n-1} \cap A_n^c) \\
= \quad p.p(1 - p)\ldots p(1 - p) \\
= \quad p^y(1 - p)^{n-y}
\]
Thus a particular sequence of $n$ trials with exactly $y$ successes has probability $p^y(1 - p)^{n-y}$ of occurring. Since there are $\binom{n}{y}$ such sequences, we have

$Y$ is a random variable that follows the Binomial distribution with two parameters $n$ and $p$, i.e., $Y \sim B(n, p)$. 
Result

(Binomial Theorem) For any real numbers $x$ and $y$ and integer $n \geq 0$,

- Setting $x = p$ and $y = 1 - p$ in the above equation,

- The binomial pmf therefore sums to one, thereby verifying the second requirement for a function to be a pmf. The first, i.e., non-negativity, follows directly from definition.
Example

Show that the mgf of $X$ is $M_{bin}(t) = [pe^t + (1 - p)]^n$, where $X \sim B(n, p)$.

Proof.

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Example
What is the probability of obtaining at least one 6 in 4 rolls of a fair die?

Solution
We can calculate this summing the binomial density values for $x = 1, 2, 3, 4$ when $n = 4$ and $p = 1/6$. (This gives the probability of at least one success in 4 die rolls). Thus, the probability we seek is:

(The last equality recognizes that the density ordinates sum to one over $x = 0, 1, \ldots, 4$).
Example
What is the probability of obtaining at least one double 6 in 24 rolls of a pair of fair dice?

Solution
Following the same argument as given in the previous example, the probability we seek is
Example (Exercise 3.1.4)

Let the independent random variables $X_1$, $X_2$ and $X_3$ have the same pdf $f(x) = 3x^2$, $0 < x < 1$. Find the probability that exactly two of these three variables exceed $1/2$.

Let $A_i$, $i = 1, 2, 3$ denote the binary event that $X_i > 1/2$. We can see that each of the $A_i$ are independently distributed Bernoulli random variables with success probability

So, we need to calculate the value of the binomial distribution with $n = 3$ and $p = 7/8$ at $x = 2$: 
The Poisson Distribution

The Poisson distribution is often used if we are modeling a phenomenon in which we are waiting for an occurrence (such as waiting for a bus, waiting for customers to arrive in a bank, etc.). The number of occurrences in a given time interval can be modeled as a Poisson random variable.

The basic assumption on which the Poisson distribution is built is that, for small time intervals, the probability of an arrival is proportional to the length of waiting time.

\[ X \sim \text{Poisson}(\lambda) \text{ if } \]
Recall the series expansion representation of $e^y$ (resulting from a Taylor series expansion of the exponential around zero):

$$e^y = \sum_{i=0}^{\infty} \frac{y^i}{i!}$$

Then so that the Poisson is a proper pmf (non-negativity follows immediately).
Similarly, 

where $y \equiv x - 1$.

Similar calculations show $\text{Var}(X) = \lambda$. 
Example (Poisson Example #1)

A typesetter, on the average, makes one error in every 500 words typeset. A typical page contains 300 words. Assuming a poisson distribution governs the number of errors in 5 pages, what is the probability that there will be no more than 2 errors in 5 average pages?

Here, the mean is $\mu = 3$, since this is the expected number of errors among 1500 words. The Poisson structure fills in the rest of what we need to know. Specifically, the probability we seek is:
Example (3.2.10)

On average, a grocer sells three of a certain article per week. How many of these should he have in stock so that the chance of his running out within a week is less than .01? Assume a Poisson distribution.

We want to find the smallest integer $n$ such that

Here are cdf values:

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<th>1</th>
<th>2</th>
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<td>.423</td>
<td>.647</td>
<td>.815</td>
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<td>.967</td>
<td>.988</td>
<td>.996</td>
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Thus, $n = 8$ is sufficient to set the chance of running out less than 1 percent.
Show the mgf of the Poisson distribution is $M_X(t) = e^{\lambda(e^t-1)}$.

Note
Prove the property that independent Poisson random variables are closed under addition. That is, if each $X_i \sim \text{Poiss}(\lambda_i)$, and the $X_i$ are independent, then

$$Y = \sum_i X_i \sim \text{Poisson}(\sum_i \lambda_i).$$

$$M_Y(t) = E(\exp(tY))$$

$$= \prod_{i=1}^n E(\exp(tY_i)) \quad \text{by independence}$$

$$= \prod_{i=1}^n \exp(\lambda_i[\exp(t) - 1])$$

$$= \exp(\sum_i \lambda_i[\exp(t) - 1])$$

$$= \exp([\sum_i \lambda_i][\exp(t) - 1])$$

which is recognized as the MGF of a $\text{Poiss}(\sum_i \lambda_i)$ random variable.
The simplest example of a continuous random variable is the uniformly distributed random variable:

\[ f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases} \]

It follows easily that \( f(x) \geq 0 \) and \( \int_a^b f(x) = 1 \).

\[ E(X) = \int_a^b \frac{x}{b-a} \, dx = \frac{b+a}{2} \]

\[ Var(X) = \frac{(b-a)^2}{12} \]
Example

Suppose that two random variables, $X_1$ and $X_2$, are uniformly distributed over the unit interval. Consider the variable $Z_2 = X_1 + X_2$. Is $Z_2$ also uniformly distributed?

First, note that

Let us then consider the joint distribution of $Z_1 = X_1$ and $Z_2$. We will ultimately integrate out $Z_1$ to obtain the desired marginal density.

By a change of variables (noting that the Jacobian of the transformation is unity), we obtain:
To obtain the marginal for $Z_2$, we must integrate out $Z_1$. The (conditional) support of $Z_1$ is a function of $Z_2$, as shown in the following graph:
Thus, for $0 \leq z_2 \leq 1$, we have:

Likewise, when $1 < z_2 \leq 2$, we have:

Putting all of this together,

$$p(z_2) = \begin{cases} 
  z_2 & \text{for } 0 \leq z_2 \leq 1 \\
  2 - z_2 & \text{for } 1 < z_2 \leq 2 \\
  0 & \text{otherwise}
\end{cases}$$

so the sum of two uniform random variables is not uniformly distributed.
Derive the moment generating function for a uniformly distributed random variable.

We seek
The pdf of the Gamma distribution is given by, with $\alpha, \beta > 0$,

$$f(x|\alpha, \beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

where $\Gamma(\alpha) \equiv \int_0^\infty t^{\alpha-1} e^{-t} dt$ is called the gamma function.
**Special cases of Gamma Distribution**

1. **Chi-square Distribution**: Set $\alpha = p/2$, $p$ an integer and $\beta = 2$:

   $$f(x|p) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2}, \quad 0 < x < \infty$$

   This is the $\chi^2$ pdf with $p$ degrees of freedom. Here $E(X) = p$ and $Var(X) = 2p$.

2. **Exponential Distribution**: Set $\alpha = 1$:

   $$f(x|\beta) = \frac{1}{\beta} e^{-x/\beta}, \quad 0 < x < \infty$$

   Here $E(X) = \beta$ and $Var(X) = \beta^2$. 
**Gamma Distribution Moment Generating Function**

\[
M_{\text{Gamma}}(t) = \int_0^\infty \exp[tx] \left[ \Gamma(\alpha) \beta^\alpha \right]^{-1} x^{\alpha-1} \exp(-x\beta^{-1}) \, dx
\]

\[
= \left[ \Gamma(\alpha) \beta^\alpha \right]^{-1} \int_0^\infty x^{\alpha-1} \exp[-x(1 - \beta t)/\beta] \, dx
\]

Now, let (for \( t < 1/\beta \))

\[
y = x(1 - \beta t)/\beta \Rightarrow x = \beta y/(1 - \beta t) \Rightarrow dx = \beta/(1 - \beta t) \, dy.
\]
Noting that the limits of integration stay constant, we can write:

\[ M_{\text{Gamma}}(t) = \int_0^\infty \frac{\beta}{1 - \beta t} [\Gamma(\alpha)\beta^\alpha]^{-1} \left( \frac{\beta y}{1 - \beta t} \right)^{\alpha - 1} \exp(-y) dy \]

\[ = \left( \frac{1}{1 - \beta t} \right)^\alpha [\Gamma(\alpha)]^{-1} \int_0^\infty y^{\alpha - 1} \exp(-y) dy \]

\[ = \left( \frac{1}{1 - \beta t} \right)^\alpha [\Gamma(\alpha)]^{-1} \Gamma(\alpha) \]

\[ = \left( \frac{1}{1 - \beta t} \right)^\alpha, \quad t < \beta^{-1} \]
From here, we obtain

\[ \mu \equiv M'(0) = \alpha \beta \]

and

\[ \sigma^2 = M''(0) - \mu^2 = \alpha(\alpha + 1)\beta^2 - \alpha^2 \beta^2 = \alpha \beta^2. \]
The Normal Distribution

- $X \sim N(\mu, \sigma^2)$ if its pdf is given by
  \[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty
  \]

- $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ is called a standard normal random variable. We have
  \[
  E(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{-\frac{z^2}{2}} \, dz = 0
  \]
  \[
  \Rightarrow E(X) = E(\mu + \sigma Z) = \mu
  \]
Similarly, \( Var(Z) = 1 \) so that

\[
Var(X) = Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2
\]

The mgf of \( X \) is expressed as

\[
M_X(t) = E[e^{tX}] = E[e^{t(\sigma Z + \mu)}] \\
= e^{\mu t} E[e^{t \sigma Z}] \\
= e^{\mu t} e^{t^2 \sigma^2 / 2} \\
= e^{\{\mu t + \frac{1}{2} \sigma^2 t^2\}}
\]
A Few Properties

1. The normal pdf is symmetric around $x = \mu$.

2. The normal pdf has a maximum of $\frac{1}{\sigma \sqrt{2\pi}}$ at $x = \mu$.

3. The inflection points (where the curve changes from convex to concave and vice-versa) are $\mu \pm \sigma$. 
Normal Quantiles can also be computed based on $Z$:

By symmetry of $Z$ around zero,

$$\Phi(-Z) = 1 - \Phi(Z)$$
Example

Let $X \sim N(2, 25)$. Find $P(0 < X < 10)$ and $P(-8 < X < 1)$. 

For the first, note:

$$
Pr(0 < X < 10) = Pr(0 - 2 < X - 2 < 10 - 2) = Pr(-2/5 < (X - 2)/5 < 8/5) = Pr(-2/5 < Z < 8/5).
$$

We can calculate this as

$$
\Phi(8/5) - \Phi(-2/5) = \Phi(8/5) - (1 - \Phi(2/5)) \approx \Phi(1.6) - 1 + \Phi(0.4) 
\approx .945 - 1 + .655 = .6.
$$
And the second
If \( X \sim N(\mu, \sigma^2), \sigma^2 > 0 \), then \( V = \frac{(X-\mu)^2}{\sigma^2} \sim \chi^2_1 \)

We will sketch a proof of a special case for simplicity (that of a single standard normal random variable). To this end, let

\[ X \sim \mathcal{N}(0, 1) \]

and define

\[ Z = X^2. \]

Note

\[
\Pr(Z \leq c) = \Pr(X^2 \leq c) = \Pr(-\sqrt{c} \leq X \leq \sqrt{c}) = \int_{-\sqrt{c}}^{\sqrt{c}} \phi(x) \, dx
\]

with \( \phi(\cdot) \) denoting the standard normal density.
Before proceeding, it is helpful to introduce two things. First, Leibnitz’s rule for differentiating integrals:

\[
\frac{\partial}{\partial t} \int_{a(t)}^{b(t)} f(x, t)dx = b'(t)f[b(t), t] - a'(t)f[a(t), t] + \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t} dx.
\]

Second, a well-known property of the Gamma function:

\[
\Gamma(1/2) = \sqrt{\pi}.
\]

Applying Leibnitz’s rule, then, to our previous expression, we obtain

Since the normal distribution is symmetric about zero, we can write this as:

\[
f_Z(c) = c^{-1/2} \phi(\sqrt{c}).
\]
Writing this out in full detail,

\[ f_Z(c) = \frac{1}{\sqrt{2\sqrt{\pi}}} c^{-1/2} \exp \left( -\frac{1}{2} c \right) \]

which is

\[ f_Z(c) = \frac{1}{2^{1/2} \Gamma[1/2]} c^{-1/2} \exp \left( -\frac{1}{2} c \right), \]

a \( \chi^2(1) \) density function.
Result

Let $X_1, \ldots, X_n$ be independent r.v.s such that, for $i = 1, \ldots, n$, $X_i \sim N(\mu_i, \sigma_i^2)$. Let $Y = \sum_{i=1}^n a_i X_i$, where $a_i, \ldots, a_n$ are constants. Then $Y \sim N(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2)$.

Result

Let $X_1, \ldots, X_n$ be i.i.d. r.v.s with a common $N(\mu, \sigma^2)$ distribution. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$. Then $\bar{X} \sim N(\mu, \sigma^2/n)$. Take $a_i = 1/n$, $\mu_i = \mu$, $\sigma_i^2 = \sigma^2$ for $i = 1, \ldots, n$ in the result above.