(1) First, note
\[ f(x, y) = f(y|x) f(x). \]

Further, observe for any \( x, y \) such that \( f(x), f(y) > 0 \):
\[
\int \frac{f(y|x)}{f(x|y)} dy = \int \left[ \frac{f(x, y)}{f(y)} \right] \left[ \frac{f(x|y)}{f(x)} \right]^{-1} dy = \int \frac{f(y)}{f(x)} dy = [f(x)]^{-1} \int f(y) dy = [f(x)]^{-1}.
\]

Note, in the first line that the joint density \( f(x, y) > 0 \) given the positivity condition, and thus we avoid problems of indeterminancy of the ratio \( f(x, y)/f(x, y) \) since the joint is non-zero. We can sub this result into our very first equation to yield:
\[
f(x, y) = f(y|x) \left[ \int \frac{f(y|x)}{f(x|y)} dy \right]^{-1}.
\]

Thus, the conditionals do define the joint (under the positivity condition). A generalization of this result is known as the 
\textit{Hammersley-Clifford Theorem}.

(2) (a) Note that, for \( 0 < y_1 < 1, y_3 > 0 \),
\[
f(y_1, y_3) = \int_0^1 (y_1 + y_2) \exp(-y_3) dy_2 = \exp(-y_3) \int_0^1 (y_1 + y_2) dy_2 = \exp(-y_3) \left[ y_2 y_2 + \frac{1}{2} y_2^2 \right]_{y_2=0}^{y_2=1} = \exp(-y_3) [y_1 + (1/2)]
\]

Therefore, we can write
\[
f(y_1, y_3) = \underbrace{\exp(-y_3) I(y_3 > 0)}_{g(y_3)} \underbrace{(y_1 + (1/2)) I(0 < y_1 < 1)}_{h(y_1)}
\]
where \( I(\cdot) \) is simply a \textit{indicator function}, equal to one if the event in parenthesis is true, and is otherwise zero. The indicator functions simply denote the support of the distribution. Thus, the joint (bivariate) distribution factors into a product of marginals, and hence \( y_1 \) and \( y_3 \) are independent.

(b) By similar reasoning, note, for \( 0 < y_1, y_2 < 1 \),

\[
f(y_1, y_2) = \int_0^\infty (y_1 + y_2) \exp(-y_3) dy_3 = (y_1 + y_2).
\]

From here, we can get the marginal:

\[
f(y_2) = \int_0^1 (y_1 + y_2) dy_1 = \left[(1/2)y_1^2 + y_2 y_1\right]_{y_1=0}^{y_1=1} = (1/2) + y_2, \quad 0 < y_2 < 1.
\]

Therefore, for \( 0 < y_2 < 1 \),

\[
f(y_1|y_2) = \frac{y_1 + y_2}{y_2 + (1/2)}, \quad 0 < y_1 < 1.
\]

Clearly this conditional is a function of \( y_2 \) and therefore not equal to the marginal of \( y_1 \), and thus \( Y_1 \) and \( Y_2 \) are \textit{NOT} independent.

(2.1.6)

\[
P(Z \leq z) = \int_0^z \int_0^{z-x} \exp(-x - y) dy dx
\]

\[
= \int_0^z \left[ \exp(-x - y) \right]_{y=0}^{z-x} dx
\]

\[
= \int_0^z \left[ \exp(-z) + \exp(-x) \right]_{x=0}^{2} dx
\]

\[
= -x \cdot \exp(-z) - \exp(-x) \bigg|_{x=0}^{2} = 1 - (z + 1) \cdot \exp(-z)
\]

\[
P(Z \leq 6) = 1 - 7 \cdot \exp(-6) \approx 0.98
\]

\[
P(Z \leq 0) = 0
\]
(2.1.7) \[ G(z) = 1 - \int_{z}^{1} \int_{y}^{1} 1 \, dy \, dx \]
\[ = 1 - \int_{z}^{1} y^{1} \left. dx \right|_{y=\frac{1}{2}} \]
\[ = 1 - \int_{z}^{1} 1 - \frac{z}{x} \, dx \]
\[ = 1 - \left( x - z \cdot \ln x \right)_{x=z}^{1} \]
\[ = z - z \cdot \ln z \]

\[ g(z) = G'(z) = \begin{cases} -\ln(z) & \text{if } 0 < z < 1, \\ 0 & \text{elsewhere} \end{cases} \]

(2.2.3) We have \( y_1 = 2x_1, y_2 = x_2 - x_1, x_1 = \frac{y_1}{2}, x_2 = y_2 + \frac{y_1}{2} \), which has the Jacobian

\[ J = \begin{bmatrix} \frac{3}{2} & 1 \\ 0 & 1 \end{bmatrix} \]

Note the support of \((X_1, X_2)\) is the set \( S = 0 < x_1 < x_2 < \infty \). Thus we have
\( 0 < \frac{y_1}{2}, \frac{y_2}{2} < y_2 + \frac{y_1}{2}, y_2 + \frac{y_1}{2} < \infty \) and obtain the support of \((Y_1, Y_2)\), \( T = 0 < y_1, y_2 < \infty \).

Therefore, the joint pdf is:
\[ f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \exp(-y_1 - y_2) & \text{if } 0 < y_1, y_2 < \infty, \\ 0 & \text{elsewhere} \end{cases} \]

(2.2.5) (a) Since \( x_1 = y_1 - y_2, x_2 = y_2 \), the Jacobian is

\[ J = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \]

with the support, \( T = -\infty < y_1, y_2 < \infty \). So, the pdf is
\[ f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 - y_2, y_2) \]

(b) Therefore, the marginal pdf of \( Y_1 \) is:
\[ f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1, X_2}(y_1 - y_2, y_2) \, dy_2 \]