33. Let $X$ be a random variable with probability density

$$f(x) = \begin{cases} c(1 - x^2), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of $c$?
(b) What is the cumulative distribution function of $X$?

34. Let the probability density of $X$ be given by

$$f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of $c$?
(b) $P \left\{ \frac{1}{2} < X < \frac{3}{2} \right\} =$ ?

35. The density of $X$ is given by

$$f(x) = \begin{cases} 10/x^2, & \text{for } x > 10 \\ 0, & \text{for } x \leq 10 \end{cases}$$

What is the distribution of $X$? Find $P\{X > 20\}$.

36. A point is uniformly distributed within the disk of radius 1. That is, its density is

$$f(x, y) = C, \quad 0 \leq x^2 + y^2 \leq 1$$

Find the probability that its distance from the origin is less than $x$, $0 \leq x \leq 1$.

37. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each having a uniform distribution over $(0,1)$. Let $M = \max(X_1, X_2, \ldots, X_n)$. Show that the distribution function of $M$, $F_M(\cdot)$, is given by

$$F_M(x) = x^n, \quad 0 \leq x \leq 1$$

What is the probability density function of $M$?

38. If the density function of $X$ equals

$$f(x) = \begin{cases} ce^{-2x}, & 0 < x < \infty \\ 0, & x < 0 \end{cases}$$

find $c$. What is $P\{X > 2\}$?

39. The random variable $X$ has the following probability mass function

$$p(1) = \frac{1}{2}, \quad p(2) = \frac{1}{3}, \quad p(24) = \frac{1}{6}$$

Calculate $E[X]$. 
40. Suppose that two teams are playing a series of games, each of which is independently won by team A with probability \( p \) and by team B with probability \( 1 - p \). The winner of the series is the first team to win four games. Find the expected number of games that are played, and evaluate this quantity when \( p = 1/2 \).

41. Consider the case of arbitrary \( p \) in Exercise 29. Compute the expected number of changeovers.

42. Suppose that each coupon obtained is, independent of what has been previously obtained, equally likely to be any of \( m \) different types. Find the expected number of coupons one needs to obtain in order to have at least one of each type.

**Hint:** Let \( X \) be the number needed. It is useful to represent \( X \) by

\[
X = \sum_{i=1}^{m} X_i
\]

where each \( X_i \) is a geometric random variable.

43. An urn contains \( n + m \) balls, of which \( n \) are red and \( m \) are black. They are withdrawn from the urn, one at a time and without replacement. Let \( X \) be the number of red balls removed before the first black ball is chosen. We are interested in determining \( \text{E}[X] \). To obtain this quantity, number the red balls from 1 to \( n \). Now define the random variables \( X_i, i = 1, \ldots, n \), by

\[
X_i = \begin{cases} 
1, & \text{if red ball } i \text{ is taken before any black ball is chosen} \\
0, & \text{otherwise}
\end{cases}
\]

(a) Express \( X \) in terms of the \( X_i \).
(b) Find \( \text{E}[X] \).

44. In Exercise 43, let \( Y \) denote the number of red balls chosen after the first but before the second black ball has been chosen.

(a) Express \( Y \) as the sum of \( n \) random variables, each of which is equal to either 0 or 1.
(b) Find \( \text{E}[Y] \).
(c) Compare \( \text{E}[Y] \) to \( \text{E}[X] \) obtained in Exercise 43.
(d) Can you explain the result obtained in part (c)?

45. A total of \( r \) keys are to be put, one at a time, in \( k \) boxes, with each key independently being put in box \( i \) with probability \( p_i \), \( \sum_{i=1}^{k} p_i = 1 \). Each time a key is put in a nonempty box, we say that a collision occurs. Find the expected number of collisions.
46. If $X$ is a nonnegative integer valued random variable, show that

$$ E[X] = \sum_{n=1}^{\infty} P\{X \geq n\} = \sum_{n=0}^{\infty} P\{X > n\} $$

**Hint:** Define the sequence of random variables $I_n$, $n \geq 1$, by

$$ I_n = \begin{cases} 
1, & \text{if } n \leq X \\
0, & \text{if } n > X 
\end{cases} $$

Now express $X$ in terms of the $I_n$.

*47. Consider three trials, each of which is either a success or not. Let $X$ denote the number of successes. Suppose that $E[X] = 1.8$.

(a) What is the largest possible value of $P\{X = 3\}$?

(b) What is the smallest possible value of $P\{X = 3\}$?

In both cases, construct a probability scenario that results in $P\{X = 3\}$ having the desired value.

48. If $X$ is uniformly distributed over $(0,1)$, calculate $E[X^2]$.

*49. Prove that $E[X^2] \geq (E[X])^2$. When do we have equality?

50. Let $c$ be a constant. Show that

(i) $\text{Var}(cX) = c^2 \text{Var}(X)$;

(ii) $\text{Var}(c + X) = \text{Var}(X)$.

51. A coin, having probability $p$ of landing heads, is flipped until head appears for the $r$th time. Let $N$ denote the number of flips required. Calculate $E[N]$.

**Hint:** There is an easy way of doing this. It involves writing $N$ as the sum of $r$ geometric random variables.

52. (a) Calculate $E[X]$ for the maximum random variable of Exercise 37.

(b) Calculate $E[X]$ for $X$ as in Exercise 33.

(c) Calculate $E[X]$ for $X$ as in Exercise 34.

53. If $X$ is uniform over $(0,1)$, calculate $E[X^n]$ and $\text{Var}(X^n)$.

54. Let $X$ and $Y$ each take on either the value 1 or $-1$. Let

$$ p(1, 1) = P\{X = 1, Y = 1\}, $$

$$ p(1, -1) = P\{X = 1, Y = -1\}, $$

$$ p(-1, 1) = P\{X = -1, Y = 1\}, $$

$$ p(-1, -1) = P\{X = -1, Y = -1\} $$
Suppose that \( E[X] = E[Y] = 0 \). Show that

(a) \( p(1, 1) = p(-1, -1) \);
(b) \( p(1, -1) = p(-1, 1) \).

Let \( p = 2p(1, 1) \). Find

(c) \( \text{Var}(X) \);
(d) \( \text{Var}(Y) \);
(e) \( \text{Cov}(X, Y) \).

55. Let \( X \) be a positive random variable having density function \( f(x) \). If \( f(x) \leq c \) for all \( x \), show that, for \( a > 0 \),

\[
P\{X > a\} \geq 1 - ac
\]

56. There are \( n \) types of coupons. Each newly obtained coupon is, independently, type \( i \) with probability \( p_i, i = 1, \ldots, n \). Find the expected number and the variance of the number of distinct types obtained in a collection of \( k \) coupons.

57. Suppose that \( X \) and \( Y \) are independent binomial random variables with parameters \( (n, p) \) and \( (m, p) \). Argue probabilistically (no computations necessary) that \( X + Y \) is binomial with parameters \( (n + m, p) \).

58. An urn contains \( 2n \) balls, of which \( r \) are red. The balls are randomly removed in \( n \) successive pairs. Let \( X \) denote the number of pairs in which both balls are red.

(a) Find \( E[X] \).
(b) Find \( \text{Var}(X) \).

59. Let \( X_1, X_2, X_3, \) and \( X_4 \) be independent continuous random variables with a common distribution function \( F \) and let

\[
p = P\{X_1 < X_2 > X_3 < X_4\}
\]

(a) Argue that the value of \( p \) is the same for all continuous distribution functions \( F \).
(b) Find \( p \) by integrating the joint density function over the appropriate region.
(c) Find \( p \) by using the fact that all 4! possible orderings of \( X_1, \ldots, X_4 \) are equally likely.

60. Calculate the moment generating function of the uniform distribution on \((0, 1)\). Obtain \( E[X] \) and \( \text{Var}[X] \) by differentiating.

61. Suppose that \( X \) takes on each of the values 1, 2, 3 with probability \( \frac{1}{3} \). What is the moment generating function? Derive \( E[X] \), \( E[X^2] \), and \( E[X^3] \) by differ-
(i) Compute \( P\{X = i\} \).
(ii) Let, for \( i = 1, 2, \ldots, k; \ j = 1, 2, \ldots, n \),

\[
X_i = \begin{cases} 
1, & \text{if the } i\text{th ball selected is white} \\
0, & \text{otherwise}
\end{cases}
\]

\[
Y_j = \begin{cases} 
1, & \text{if white ball } j \text{ is selected} \\
0, & \text{otherwise}
\end{cases}
\]

Compute \( E[X] \) in two ways by expressing \( X \) first as a function of the \( X_i \)s and then of the \( Y_j \)s.

*72. Show that \( \text{Var}(X) = 1 \) when \( X \) is the number of men who select their own hats in Example 2.31.

73. For the multinomial distribution (Exercise 17), let \( N_i \) denote the number of times outcome \( i \) occurs. Find

(i) \( E[N_i] \);
(ii) \( \text{Var}(N_i) \);
(iii) \( \text{Cov}(N_i, N_j) \);
(iv) Compute the expected number of outcomes that do not occur.

74. Let \( X_1, X_2, \ldots \) be a sequence of independent identically distributed continuous random variables. We say that a record occurs at time \( n \) if \( X_n > \max(X_1, \ldots, X_{n-1}) \). That is, \( X_n \) is a record if it is larger than each of \( X_1, \ldots, X_{n-1} \). Show

(i) \( P(\text{a record occurs at time } n) = 1/n \);
(ii) \( E[\text{number of records by time } n] = \sum_{i=1}^{n} 1/i \);
(iii) \( \text{Var}(\text{number of records by time } n) = \sum_{i=1}^{n} (i - 1)/i^2 \);
(iv) Let \( N = \min\{n: n > 1 \text{ and a record occurs at time } n\} \). Show \( E[N] = \infty \).

**Hint:** For (ii) and (iii) represent the number of records as the sum of indicator (that is, Bernoulli) random variables.

75. Let \( a_1 < a_2 < \cdots < a_n \) denote a set of \( n \) numbers, and consider any permutation of these numbers. We say that there is an inversion of \( a_i \) and \( a_j \) in the permutation if \( i < j \) and \( a_j \) precedes \( a_i \). For instance the permutation \( 4, 2, 1, 5, 3 \) has 5 inversions—(4, 2), (4, 1), (4, 3), (2, 1), (5, 3). Consider now a random permutation of \( a_1, a_2, \ldots, a_n \)—in the sense that each of the \( n! \) permutations is equally likely to be chosen—and let \( N \) denote the number of inversions in this permutation. Also, let

\[
N_i = \text{number of } k: k < i, \ a_i \text{ precedes } a_k \text{ in the permutation}
\]

and note that \( N = \sum_{i=1}^{n} N_i \).
2 Random Variables

(i) Show that $N_1, \ldots, N_n$ are independent random variables.
(ii) What is the distribution of $N_i$?
(iii) Compute $E[N]$ and $\text{Var}(N)$.

76. Let $X$ and $Y$ be independent random variables with means $\mu_x$ and $\mu_y$ and variances $\sigma_x^2$ and $\sigma_y^2$. Show that

$$\text{Var}(XY) = \sigma_x^2 \sigma_y^2 + \mu_x^2 \sigma_y^2 + \mu_x^2 \sigma_y^2$$

77. Let $X$ and $Y$ be independent normal random variables, each having parameters $\mu$ and $\sigma^2$. Show that $X + Y$ is independent of $X - Y$.

Hint: Find their joint moment generating function.

78. Let $\phi(t_1, \ldots, t_n)$ denote the joint moment generating function of $X_1, \ldots, X_n$.

(a) Explain how the moment generating function of $X_i$, $\phi_{X_i}(t_i)$, can be obtained from $\phi(t_1, \ldots, t_n)$.
(b) Show that $X_1, \ldots, X_n$ are independent if and only if

$$\phi(t_1, \ldots, t_n) = \phi_{X_1}(t_1) \cdots \phi_{X_n}(t_n)$$

References

7. Suppose \( p(x, y, z) \), the joint probability mass function of the random variables \( X, Y, \) and \( Z \), is given by

\[
\begin{align*}
    p(1, 1, 1) &= \frac{1}{8}, & p(2, 1, 1) &= \frac{1}{4}, \\
    p(1, 1, 2) &= \frac{1}{8}, & p(2, 1, 2) &= \frac{3}{16}, \\
    p(1, 2, 1) &= \frac{1}{16}, & p(2, 2, 1) &= 0, \\
    p(1, 2, 2) &= 0, & p(2, 2, 2) &= \frac{1}{4}
\end{align*}
\]

What is \( E[X|Y = 2] \)? What is \( E[X|Y = 2, Z = 1] \)?

8. An unbiased die is successively rolled. Let \( X \) and \( Y \) denote, respectively, the number of rolls necessary to obtain a six and a five. Find (a) \( E[X] \), (b) \( E[X|Y = 1] \), (c) \( E[X|Y = 5] \).

9. Show in the discrete case that if \( X \) and \( Y \) are independent, then

\[
E[X|Y = y] = E[X] \quad \text{for all } y
\]

10. Suppose \( X \) and \( Y \) are independent continuous random variables. Show that

\[
E[X|Y = y] = E[X] \quad \text{for all } y
\]

11. The joint density of \( X \) and \( Y \) is

\[
f(x, y) = \frac{(y^2 - x^2)}{8} e^{-y}, \quad 0 < y < \infty, \quad -y \leq x \leq y
\]

Show that \( E[X|Y = y] = 0 \).

12. The joint density of \( X \) and \( Y \) is given by

\[
f(x, y) = \frac{e^{-x/y} e^{-y}}{y}, \quad 0 < x < \infty, \quad 0 < y < \infty
\]

Show \( E[X|Y = y] = y \).

*13. Let \( X \) be exponential with mean \( 1/\lambda \); that is,

\[
f_X(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty
\]

Find \( E[X|X > 1] \).

14. Let \( X \) be uniform over \((0, 1)\). Find \( E[X|X < \frac{1}{2}] \).

15. The joint density of \( X \) and \( Y \) is given by

\[
f(x, y) = \frac{e^{-y}}{y}, \quad 0 < x < y, \quad 0 < y < \infty
\]

Compute \( E[X^2|Y = y] \).
16. The random variables $X$ and $Y$ are said to have a bivariate normal distribution if their joint density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \times \left[ \left( \frac{x-\mu_x}{\sigma_x} \right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left( \frac{y-\mu_y}{\sigma_y} \right)^2 \right] \right\}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, where $\sigma_x$, $\sigma_y$, $\mu_x$, $\mu_y$, and $\rho$ are constants such that $-1 < \rho < 1$, $\sigma_x > 0$, $\sigma_y > 0$, $-\infty < \mu_x < \infty$, $-\infty < \mu_y < \infty$.

(a) Show that $X$ is normally distributed with mean $\mu_x$ and variance $\sigma_x^2$, and $Y$ is normally distributed with mean $\mu_y$ and variance $\sigma_y^2$.

(b) Show that the conditional density of $X$ given that $Y = y$ is normal with mean $\mu_x + (\rho\sigma_x/\sigma_y)(y - \mu_y)$ and variance $\sigma_x^2(1-\rho^2)$.

The quantity $\rho$ is called the correlation between $X$ and $Y$. It can be shown that

$$\rho = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y} = \frac{\text{Cov}(X, Y)}{\sigma_x\sigma_y}$$

17. Let $Y$ be a gamma random variable with parameters $(s, \alpha)$. That is, its density is

$$f_Y(y) = Ce^{-\alpha y}y^{s-1}, \quad y > 0$$

where $C$ is a constant that does not depend on $y$. Suppose also that the conditional distribution of $X$ given that $Y = y$ is Poisson with mean $y$. That is,

$$P\{X = i|Y = y\} = e^{-y}y^i/i!, \quad i \geq 0$$

Show that the conditional distribution of $Y$ given that $X = i$ is the gamma distribution with parameters $(s + i, \alpha + 1)$.

18. Let $X_1, \ldots, X_n$ be independent random variables having a common distribution function that is specified up to an unknown parameter $\theta$. Let $T = T(X)$ be a function of the data $X = (X_1, \ldots, X_n)$. If the conditional distribution of $X_1, \ldots, X_n$ given $T(X)$ does not depend on $\theta$ then $T(X)$ is said to be a sufficient statistic for $\theta$. In the following cases, show that $T(X) = \sum_{i=1}^n X_i$ is a sufficient statistic for $\theta$. 