## EAPS 53600: Intro to General Circulation of the Atmosphere <br> HW 03: Wave theory

In class we have shown theoretically how a single shallow layer of fluid of constant density can exhibit various types of waves. Specifically, we have considered this fluid 1) without background rotation $(f=0) ; 2)$ with constant background rotation $\left(f=f_{0}\right)$; and 3 ) with varying background rotation $\left(f=f_{0}+\beta y\right)$.

These waves are also found in stratified fluids (with variable density), such as our atmosphere and ocean. In such cases, the waves are conceptually identical, but the math is a bit more complex (including accounting for vertical scale/propagation).

We first need to become more comfortable with the theories derived in class. Specifically, theory has defined a small number of physical parameters whose values depend on the properties of the fluid. The structure and dynamics of these waves are hypothesized to depend on these parameter values. Thus, it is important to first think about these parameters themselves and the equations that govern the behavior of the waves as applied to the Earth.

## 1 Fundamental physical parameters

## 1. Rotation

(a) Planetary vorticity: $f=2 \Omega \sin \phi$
i. Calculate and plot $f$ vs. latitude on Earth (every 10 degrees from -90 to 90)
ii. At which latitude is $f$ largest? Zero? Smallest?
(b) Planetary vorticity gradient: $\beta=\frac{2 \Omega}{a} \cos \phi$
i. Calculate and plot $\beta$ vs. latitude on Earth (every 10 degrees from -90 to 90)
ii. At which latitude is $\beta$ largest? Zero? Smallest?
2. Stratification: gravity wave phase speed
(a) Single layer of shallow fluid with a free upper surface $\left(c_{G W}=\sqrt{g H}\right)$
i. Calculate and plot the non-rotating gravity wave phase speed vs. fluid depth $(10 \mathrm{~cm}, 1 \mathrm{~m}, 10 \mathrm{~m}, 100 \mathrm{~m}, 1 \mathrm{~km}$, and 10 km$)$.
ii. At which depth is $c_{G W}$ largest? Smallest?
(b) For the Earth's stratified atmosphere $\left(c_{G W}=N H\right)$
i. Calculate and plot the tropospheric non-rotating gravity wave phase speed vs. latitude (same latitudes as previous problem). To estimate parameters, use the data shown in Figure 1 below. You may define $H$ to be the depth
of the troposphere and $N$ and as average value over the depth of the troposphere. Specifically, $N=\sqrt{\frac{g}{\theta_{0}} \frac{\partial \theta}{\partial z}}$, where $\theta_{0}$ is the tropospheric-mean potential temperature and $\frac{\partial \theta}{\partial z} \approx \frac{\Delta \theta}{H}$, where $\Delta \theta$ is the bulk change in $\theta$ over the depth $H$.
ii. At which latitude is $c_{G W}$ largest? Smallest?
(c) In class we discussed how the dynamics of the stratified atmosphere in pressure coordinates (sloping pressure surface) is directly analogous to the one-layer shallow water system (sloping fluid top surface). Suppose the real atmosphere behaved like (i.e. had a gravity wave phase speed equal to) a one-layer shallow water system with some equivalent depth, $h_{e q}$, in lieu of $H$. (Note: this concept is also sometimes defined as a reduced gravity, $g^{\prime}$ )
i. How deep would our equivalent one-layer atmosphere be? Using your results from the previous problem, calculate and plot $h_{e q}$ vs. latitude (same latitudes as previous problem).
ii. Does the equivalent depth slope upwards or downwards moving poleward?
3. Combined rotation + stratification: Rossby deformation radius ( $L_{d}=\frac{c_{G W}}{f}$ )
(a) Using your answers for the Earth's troposphere, calculate and plot the tropospheric Rossby deformation radius vs. latitude.
(b) At which latitude is $L_{d}$ largest? Smallest? Infinite?
(c) For Earth, a change in one degree latitude equates to a distance of 111.325 km . Over what range of latitudes is $L_{d}$ larger than the distance to the Equator?
4. The deep tropics. Near the equator, the Rossby deformation radius becomes very large. Thus, a second, very similar deformation length-scale becomes relevant called the equatorial deformation radius, given by $L_{d, E Q}=\sqrt{\frac{c_{G W}}{\beta}}$. This is the intrinsic length scale specific to the equatorial beta-plane.
(a) Derive this expression for $L_{d, E Q}$. Start with the equation for $L_{d}$ and note that near the equator $f_{0}=0$ and so $f \approx \beta L$, where $L$ is the meridional distance from the equator.
(b) Using your answers for the Earth's troposphere, what is the equatorial deformation radius on Earth (i.e. at $\phi=0$ )? To what latitude north/south of the equator does this length scale extend?
5. The flat tropical atmosphere. Figure 2 shows the annual-mean 500 hPa geopotential height and temperature for 2016 . The 500 hPa surface is representative of the middle/upper free troposphere.
(a) Qualitatively, how does the slope of the 500 hPa pressure surface vary with latitude in lower vs. higher latitudes?
(b) Qualitatively, how does magnitude of the horizontal temperature gradient vary with latitude in lower vs. higher latitudes?
(c) Using your earlier answers for the deformation radius and your physical understanding of this length scale, explain why the tropical free troposphere must be relatively uniform horizontally. (Note: this underlies a common assumption made when modeling the tropical atmosphere called the Weak Temperature Gradient approximation or Weak Pressure Gradient approximation; for more information, see Sobel and Bretherton (2000, JAS) or Romps (2012, JAS).)


Figure 1: Zonal-mean potential temperature ([K]; thin black) and tropopause (thick black) for January. Altitudes are only approximate. Source: Mahlman (1997, Science) https://science.sciencemag.org/content/276/5315/1079/tab-figures-data.


Figure 2: Left: annual-mean (geopotential) height of 500 hPa surface for 2016. Right: annual-mean 500 hPa temperature for 2016. Source: NCEP/NCAR Reanalysis. (Note: you can make these types of plots yourself very quickly here: https://www.esrl.noaa.gov/psd/data/composites/day/.)

## 2 Wave properties

1. Plot the dispersion relation ( $\omega$ vs. $k$ ) for each of the following, with wavenumber along the x -axis and wave frequency along the y -axis. Following convention, $k<0$
corresponds to westward propagation, $k>0$ corresponds to eastward propagation, and $\omega \geq 0$.
(a) Non-rotating gravity waves, for fluid depths of $H=10 \mathrm{~m}, 100 \mathrm{~m}$, and 1 km .
(b) Rotating gravity waves (a.k.a. "Inertio-gravity" or "Poincare" waves) at latitude 40 N , for fluid depths of $H=10 \mathrm{~m}, 100 \mathrm{~m}$, and 1 km
(c) Rossby waves at latitude 40 N , with zero mean flow and for deformation radii of $L_{d}=100 \mathrm{~km}, 1000 \mathrm{~km}$, and $\infty$. Assume $l=0$ (i.e. uniform in meridional direction). (Recall: $k_{d}=\frac{1}{L_{d}}$.)
2. Calculate 1) the phase velocity and 2) the group velocity for a Rossby wave with zero mean flow, infinite deformation radius and...
(a) wavelengths $\left(\lambda_{x}, \lambda_{y}\right)=(300 \mathrm{~km}, 300 \mathrm{~km})$ at 10 N
(b) wavelengths $\left(\lambda_{x}, \lambda_{y}\right)=(3000 \mathrm{~km}, 3000 \mathrm{~km})$ at 10 N
(c) wavelengths $\left(\lambda_{x}, \lambda_{y}\right)=(300 \mathrm{~km}, 300 \mathrm{~km})$ at 60 N
(d) wavelengths $\left(\lambda_{x}, \lambda_{y}\right)=(3000 \mathrm{~km}, 3000 \mathrm{~km})$ at 60 N
(e) Based on your results, are these waves dispersive? If yes, are they more dispersive at lower or higher latitude?
3. Calculate 1) the phase velocity and 2) the group velocity for a Rossby wave with zero mean flow and deformation radius estimated using your result from the previous problem with...
(a) wavelengths $\left(\lambda_{x}, \lambda_{y}\right)=(300 \mathrm{~km}, 300 \mathrm{~km})$ at 10 N
(b) wavelengths $\left(\lambda_{x}, \lambda_{y}\right)=(3000 \mathrm{~km}, 3000 \mathrm{~km})$ at 10 N
(c) wavelengths $\left(\lambda_{x}, \lambda_{y}\right)=(300 \mathrm{~km}, 300 \mathrm{~km})$ at 60 N
(d) wavelengths $\left(\lambda_{x}, \lambda_{y}\right)=(3000 \mathrm{~km}, 3000 \mathrm{~km})$ at 60 N
(e) Based on your results, are these waves dispersive? If yes, are they more dispersive at lower or higher latitude?
4. Calculate 1) the phase velocity and 2) the group velocity for a Kelvin wave propagating zonally along the Equator ("equatorially-trapped") with equivalent depth equal to that found in your earlier problem...
(a) zonal wavelength $\lambda_{x}=300 \mathrm{~km}$
(b) zonal wavelength $\lambda_{x}=3000 \mathrm{~km}$
(c) Based on your results, are these waves dispersive?
(d) How much time will it take for each of these waves take to circumnavigate the Earth? Which one is faster?
5. Suppose there is a source of barotropic Rossby waves (i.e. $m=0$, no vertical structure) centered at $(\lambda, \phi)=(90 E, 10 N)$. For simplicity, suppose further that this source produces waves at exactly two horizontal length scales: 1) $\lambda_{x}=1000 \mathrm{~km}$ and $\lambda_{y}=$ 1000 km ; 2) $\lambda_{x}=2000 \mathrm{~km}$ and $\lambda_{y}=2000 \mathrm{~km}$
(a) In what direction and at what speed will the energy associated with the smaller wave move? The larger wave?
(b) After 12 hours, at what latitude and longitude will the center of this wave energy be found for the smaller wave? The larger wave?
(c) Qualitatively, what do you expect the amplitude of the total perturbation of these Rossby waves to look like at this time ( 12 hours)?
