EAPS 59100, Spring 2020
Lec 03: Equations of motion for rotating fluid

Reading: VallisE Ch2

1 Basic intro

Atmosphere / ocean: shallow layers of fluid on a sphere, strongly influenced by 1) rotation and 2) stratification

- fluid: a continuum of particles that flows and deforms
- shallow: $H \ll L$
- rotation: Ω
- stratification: $\frac{\partial \overline{\rho}}{\partial z} \gg \nabla_h \overline{\rho}$

2 Equations of motion for a rotating fluid

VallisE p27

Cartesian coordinates (x,y,z):

Inviscid momentum equations in the **rotating frame of reference** (i.e. $\vec{\mathbf{v}} = (u, v, w)$ – 3D velocity for air parcels in rotating frame of reference – i.e. what we think of as the "wind"):

$\vec{\mathbf{v}} = (u, v, w)$

total (Lagrangian) derivative:

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \vec{\mathbf{v}} \cdot \nabla A = \frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + v\frac{\partial A}{\partial y} + w\frac{\partial A}{\partial z} \tag{1}$$

Momentum equations:

$$\frac{D\vec{\mathbf{v}}}{Dt} + 2\vec{\Omega} \times \vec{\mathbf{v}} = -\frac{1}{\rho}\nabla p - g\hat{\mathbf{k}} \quad [ms^{-2}] \tag{2}$$

Coriolis acceleration: $\vec{\mathbf{a}}_{Cor} = -2\vec{\Omega} \times \vec{\mathbf{v}}$

- Zero Coriolis force on body that is stationary in rotating frame
- Coriolis force deflects moving bodies at right angles to their direction of motion

3 Spherical coordinates

VallisE p28

Spherical coordinates (**FIG 2.2**):

- r: radius from planet center
- ϕ: latitude (**differs from Vallis!**)
- λ: longitude
- $\vec{\mathbf{v}} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}} = (u, v, w) = \left(rcos\phi\frac{D\lambda}{Dt}, r\frac{D\phi}{Dt}, \frac{Dr}{Dt}\right) =$ (zonal, meridional, vertical) directions

Note: $\frac{Dr}{Dt} = \frac{Dz}{Dt}$
total (Lagrangian) derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{rcos\phi}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial r} \tag{3}$$

VallisE p32

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{rcos\phi}\right)(vsin\phi - wcos\phi) = -\frac{1}{\rho rcos\phi}\frac{\partial p}{\partial \lambda} \quad [ms^{-2}] \tag{4}$$

$$\frac{Dv}{Dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{rcos\phi}\right)(usin\phi) = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} \quad [ms^{-2}] \tag{5}$$

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} - 2\Omega ucos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g \quad [ms^{-2}] \tag{6}$$

$$\tag{7}$$

Notes:

- "Coriolis" terms: terms with Ω
- "Metric" terms: LHS quadratic terms with $\frac{1}{r}$, associated with curvature of sphere (specifically that $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ directions aren't straight but curve around sphere)

4 Primitive equations

VallisE p32

Three approximations to spherical equations above:

- Hydrostatic: in vertical momentum equation, neglect 1) advection of w, 2) Coriolis, and 3) metric terms
- Shallow fluid: 1) $r \approx a$; 2) in derivatives $dr \approx dz$ (a is constant)

- Traditional: neglect terms involving w multiplied by u or v ($|w| \ll u$ or v)

Note: 2+3 = thin shell approx.

$$\frac{Du}{Dt} - fv - \frac{uv}{a} tan\phi = -\frac{1}{\rho a cos\phi}\frac{\partial p}{\partial \lambda} \quad [ms^{-2}] \tag{8}$$
$$\frac{Dv}{Dt} + fu + \frac{u^2}{a} tan\phi = -\frac{1}{\rho a}\frac{\partial p}{\partial \phi} \quad [ms^{-2}] \tag{9}$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad [ms^{-2}] \tag{10}$$
$$\tag{11}$$

with Coriolis parameter

$$f = 2\Omega sin\phi \quad [s^{-1}] \tag{12}$$

and with total (Lagrangian) derivative (note: error in book VallisE p32)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a cos\phi}\frac{\partial}{\partial \lambda} + \frac{v}{a}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial z} \tag{13}$$

5 Cartesian approximations to the primitive equations

5.1 f-plane

VallisE p32

Consider motions for which $L \ll a$ – layer of fluid approximated by tangent plane of sphere (DRAW A DIAGRAM SPHERE + tangent plane))

Geometry:

- Plane tangent to Earth's surface at latitude ϕ_0
- small excursions: $(x, y, z) \approx (a\lambda cos\phi_0, a(\phi - \phi_0), z)$, $\vec{\mathbf{v}} = (u, v, w)$ = velocity **in the tangent plane** (i.e. approximately West-East, South-North, vertical)
- constant Coriolis parameter $f_0 = 2\Omega sin\phi_0$
- No curvature: ignore LHS quadratic terms in u and v

1) Horizontal momentum equations: (note $dx = acos\phi_0 d\lambda$, $dy = ad\phi$)

$$\frac{Du}{Dt} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad [ms^{-2}] \tag{14}$$
$$\frac{Dv}{Dt} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad [ms^{-2}] \tag{15}$$
$$\tag{16}$$

(note: vertical momentum equation isn't altered – can do hydrostatic or non-hydrostatic) which may also be written in vector form as

$$\frac{D\vec{\mathbf{v}}}{Dt} + f_0\hat{\mathbf{k}} \times \vec{\mathbf{v}} = -\frac{1}{\rho}\nabla p - g\hat{\mathbf{k}} \quad [ms^{-2}] \tag{17}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\mathbf{v}} \cdot \nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{18}$$

Identical to rotating cartesian system in which the rotation vector Ω is aligned with the local vertical

f_0 represents the projection of the planetary vorticity onto the local vertical at latitude ϕ_0.

Analog? ROTATING TANK!

5.2 Beta-plane

VallisE p33

$f = 2\Omega sin\phi$ varies with latitude.

On the tangent plane, mimic this variation in y. Taylor expansion of f about f_0, i.e. $f \approx f_0 + \frac{df}{dy}y$, where $f = f_0$ at $y = 0$:

$$f \approx f_0 + \beta y \quad [s^{-1}] \tag{19}$$

where $\beta = \frac{df}{dy} = \frac{df}{ad\phi} = \frac{1}{a}\frac{df}{d\phi}$

$$\beta = \frac{2\Omega cos\phi_0}{a} \quad [m^{-1}s^{-1}] \tag{20}$$

- Captures **dynamical** effect of sphericity – ***differential*** rotation
- Avoids complicating **geometric** effects (curvature of sphere)

Equations same as f-plane above, except with $f_0 + \beta y$ in lieu of f_0.

6 Mass continuity and thermodynamic equation

VallisE p35 (and p9)

General mass continuity equation (a density budget)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{\mathbf{v}}) = 0 \tag{21}$$

(Interpretation: density will be increased ($\frac{\partial \rho}{\partial t}$) if there is a convergence of flux of density by the flow ($-\nabla \cdot (\rho\vec{\mathbf{v}})$).

General thermodynamic equation: some thermodynamic quantity S (e.g. temperature, potential temperature, entropy, buoyancy)

$$\frac{DS}{Dt} = \dot{S} \tag{22}$$

The 'dot' means sources/sinks

7 Boussinesq approximation

VallisE p36

Constant reference density, ρ_0.

$\rho(x, y, z) = \rho_0 + \rho'(x, y, z)$

Assumes that $\frac{\rho'}{\rho_0} \ll 1$ – more ocean-like.

Reference pressure distribution in hydrostatic balance with ρ_0:

$\frac{dp_0}{dz} = -\rho_0 g$ (i.e. p_0 varies linearly with height).

We **only** allow for variations of density to appear in buoyancy – gravity is large, so buoyant vertical accelerations can be large even with only small $\delta\rho$!

$p(x, y, z) = p_0(z) + p'(x, y, z)$

1) Momentum:

$$\frac{D\vec{\mathbf{v}}}{Dt} + f\hat{\mathbf{k}} \times \vec{\mathbf{v}} = -\frac{1}{\rho_0}\nabla p' + b\hat{\mathbf{k}} \quad [ms^{-2}] \tag{23}$$

with buoyancy

$$b = -g\frac{\rho'}{\rho_0} \quad [ms^{-2}] \tag{24}$$

$p_0(z)$ is in hydrostatic balance with g and thus are equal and opposite and subtract out.

2) Continuity: (now diagnostic, not prognostic; volume is constant)

$$\nabla \cdot \vec{\mathbf{v}} = 0 \quad [s^{-1}] \tag{25}$$

This does **not** mean $\frac{D\rho}{Dt} = 0$. Density of a parcel can absolutely change as it moves – volume is constant (i.e. incompressible), but **air parcels still experience buoyant vertical accelerations**. This is basically the simplest possible way to have a parcel whose density is approximately constant but which can still be accelerated up/down by buoyancy. It's weird, but math allows it.

3) Thermodynamic:

$$\frac{Db}{Dt} = \dot{b} \quad [ms^{-3}] \tag{26}$$

A parcel becomes buoyant if a source or sink or buoyancy acts on it.

8 Anelastic equations: a stratified fluid

VallisE p39

Exact same as Boussinesq, except now reference density $\tilde{\rho}(z)$ (i.e. varies with height). ("Weak Boussinesq")

$\rho(x, y, z) = \tilde{\rho}(z) + \rho'(x, y, z)$

Assumes that $\frac{\rho'}{\tilde{\rho}(z)} \ll 1$ (atmosphere-like).

End result: reference state is now $\theta_0 = constant$ (instead of $\rho_0 = constant$)

Differences in equations:

buoyancy

$$b = -g\frac{\theta'}{\theta_0} \quad [ms^{-2}] \tag{27}$$

Continuity:

$$\nabla \cdot \vec{\mathbf{u}} + \frac{1}{\tilde{\rho}}\frac{\partial}{\partial z}(\tilde{\rho}w) = 0 \quad [s^{-1}] \tag{28}$$

Horizontal velocity $\vec{\mathbf{u}} = (u, v, 0)$

$\tilde{\rho}$ varies with height and so cannot be pulled out from within the vertical derivative (and thus cancel out).