EAPS 53600, Spring 2020
Lec 04a: Shallow water dynamics

Reading: VallisE Ch 4

1 Shallow water system: basics

VallisE p63

Shallow water equations describe:

- a thin layer of fluid ($H \ll L$)
- constant density (ρ_0)
- hydrostatic balance
- lower boundary = rigid surface
- upper boundary = free surface, above which is another fluid of negligible inertia

Benefits:

- Very useful model of effects of rotation on fluid without the complicating effects of stratification.
- Fluid motion fully determined by momentum + mass continuity equations (constant density = no *thermodynamic* variation in fluid)

Note: stacking multiple such layers of immiscible fluid of different densities = stably-stratified "stacked shallow water" system. This behaves like a continuously stratified system.

2 Equations

FIG 4.1

Fluid thickness: $h(x, y, t) = \eta(x, y, t) - \eta_b(x, y, t)$
Height of bottom surface (i.e. topography): η_b, where $\overline{\eta_b} = 0$
Height of free surface: η (if flat bottomed, i.e. $\eta_b = 0$, then $\eta = h$)

1) Momentum:
Vertical: hydrostatic balance
$\frac{\partial p}{\partial z} = -\rho_0 g$
Integrate downwards from fluid surface, assume $p(x, y, \eta, t) = 0$ at top of fluid (i.e. assume overlying mass is negligible)
$\int_0^p \partial p = -\rho_0 g \int_\eta^z \partial z$

$0 - p = -\rho_0 g(\eta(x, y, t) - z)$
$p(x, y, t) = \rho_0 g(\eta(x, y, t) - z)$

Horizontal:
Horizontal velocity: $\vec{\mathbf{u}} = (u, v)$
The horizontal pressure gradient is independent of height!
$\nabla p = \rho_0 g \nabla \eta$

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} + \vec{\mathbf{u}} \cdot \nabla \vec{\mathbf{u}} + f\hat{\mathbf{k}} \times \vec{\mathbf{u}} = -g\nabla\eta \quad [ms^{-2}] \tag{1}$$

Note: the horizontal velocity $\vec{\mathbf{u}}(x, y, t)$ is also independent of height!

2) Mass (continuity):
Incompressible: $\nabla \cdot \vec{\mathbf{v}} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$
$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial y}{\partial y}\right) = -\nabla \cdot \vec{\mathbf{u}}$
Integrate upwards from the bottom to the top; $\nabla \cdot \vec{\mathbf{u}}$ is independent of height
$w(\eta) - w(\eta_b) = -h(\nabla \cdot \vec{\mathbf{u}})$
$w(\eta) = \frac{D\eta}{Dt}$ – vertical velocity at fluid top literally changes the height of the fluid top
$w(\eta_b) = \frac{D\eta_b}{Dt} = 0$ – same for fluid bottom (e.g. an earthquake)
Thus
$\frac{D}{Dt}(\eta - \eta_b) = -h\nabla \cdot \vec{\mathbf{u}}$
$\frac{Dh}{Dt} = -h\nabla \cdot \vec{\mathbf{u}}$ – moving with the flow, the height will increase if the volume increases (flow convergence).
$\frac{Dh}{Dt} + h\nabla \cdot \vec{\mathbf{u}} = 0$

$$\frac{\partial h}{\partial t} + \vec{\mathbf{u}} \cdot \nabla h + h\nabla \cdot \vec{\mathbf{u}} = 0 \quad [ms^{-1}] \tag{2}$$

Equivalently $\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{\mathbf{u}}) = 0$ – the height of the fluid increases if there is a convergent flux of height (!) by the flow.
Note: **FIG 4.2** – VallisE gives an alternative derivation

3 Balanced flow: Geostrophic balance (VallisE Ch 3.3)

Geostrophic balance:
Scale analysis of LHS of horizontal momentum equation: $\vec{\mathbf{u}} \cdot \nabla \vec{\mathbf{u}} \sim \frac{U^2}{L}$, $f\hat{\mathbf{k}} \times \vec{\mathbf{u}} \sim fU$
Ratio: Rossby number $Ro = \frac{U}{fL}$
If $Ro \ll 1$: Coriolis $\gg$ advection – can neglect advection term

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} + f\hat{\mathbf{k}} \times \vec{\mathbf{u}} = -g\nabla\eta \tag{3}$$

Geostrophic balance:

$$f\hat{\mathbf{k}} \times \vec{\mathbf{u}}_g = -g\nabla\eta \tag{4}$$

The geostrophic wind speed $|\vec{\mathbf{u}}_g|$ is proportional to the slope of the surface (FIG 3.6). This is also true on pressure coordinates (height = height of pressure surface).

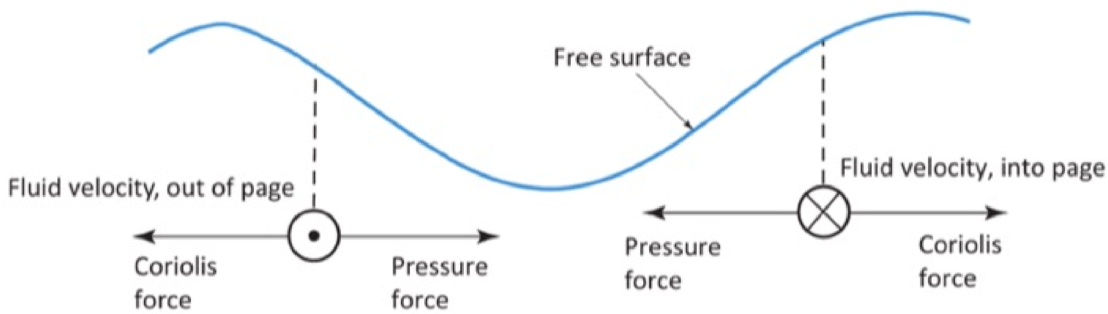

Figure 1: VallisE Fig 4.6: Geostrophic flow in one-layer shallow water system ($f > 0$).

Fig. 4.6:

Optional: Margules two-layer shallow water slope, thermal wind balance (cf. separate PDF notes)

4 Review: waves

VallisE 6.1-6.2. See box "Wave Fundamentals" on p107, as well as Figure 2 below.

4.1 Wave basics: (VallisE Fig 6.1)

- **wave**: an oscillating motion, caused by a restoring force. Can move at some speed in a direction.
- **(angular) frequency**: ω; the angular displacement per unit time (rate of change of phase of wave) at a fixed location.
- **(angular) wavenumber**: (x: k; y: l; z: m); the number of radians per unit distance. A true vector ($\vec{\mathbf{k}} = (k, l, m)$).
  - **wavevector**: $\vec{\mathbf{k}} = (k, l, m)$; the direction of propagation and its magnitude.
  - **total wavenumber**: $K^2 = k^2 + l^2 + m^2$; magnitude of the wavevector
- **(linear) wavelength**: $\lambda = \frac{2\pi}{K}$; the linear distance between two consecutive wave peaks (or troughs, or any other phase of the wave). The inverse of wavenumber. x: $\lambda_x = \frac{2\pi}{k}$; y: $\lambda_y = \frac{2\pi}{l}$; z: $\lambda_z = \frac{2\pi}{m}$ – but **NOT** a true vector (there is no "wavelength vector" $\vec{\lambda}$)!

Frequent point of confusion: we use angular wavenumber (k) in the standard equation to define a wave (see below). However, we typically think about the size of waves in terms of the *spatial* wavelength (λ). This is why we need the 2π factor when converting between the two – we care about the direct spatial distance from crest to crest, not the angular distance along the wave itself.

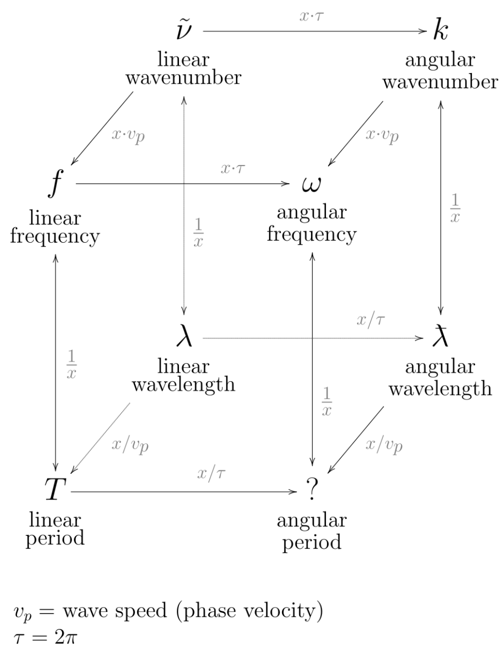

Figure 2: Relationships between frequency, wavenumber, and wavelength – their spatial vs. angular counterparts. Source: https://en.wikipedia.org/wiki/Wavenumber.

4.2 Wave motion basics: (VallisE Fig 6.2)

- **Dispersion relation**: $\omega(\mathbf{k})$; equation relating wave frequency and size (defined by the wavevector)
- **Phase speed**: $c_p = \omega/K$, $K = 2\pi/\lambda$; speed at which the wave crests move. $c_p^x = \omega/k$, $c_p^y = \omega/l$, $c_p^z = \omega/m$ – but **NOT** a true vector (there is no "phase velocity" $\vec{c}_p$)! .
- **Group velocity**: $\vec{\mathbf{c}}_g = (c_g^x, c_g^y, c_g^z) = \nabla_k \omega = (\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m})$; velocity at which a wave packet (i.e. envelope of waves *and* wave energy) moves. Most physical quantities of interest are transported at the group velocity. See VallisE 6.2.1 and Fig 6.2 to understand why the group velocity is defined as a derivative.

Notes:

- Standard convection is to define frequency (ω) to be non-negative, i.e. $\omega \geq 0$. Wavenumber (k, l, m) may be negative or positive; positive = eastward/northward/upward; negative = westward/southward/downward.
- Why? The wavevector is $\mathbf{k}\cdot\mathbf{x}$, where $\mathbf{x}$ is defined positive eastward/northward/upward. Thus $\mathbf{k}\cdot\mathbf{x} < 0$ means that e.g. for zonal motion $k < 0$ points opposite to the positive x direction (eastward) and hence the wave will move westward.

4.3 Standard equation for monochromatic plane waves

$$\boxed{\eta' = Re\{\tilde{\eta} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}} - \omega t)}\}} \tag{5}$$

Note: this can also be written as $\eta' = Re\{\tilde{\eta}e^{i(Kx^* - \omega t)}\}$, where x^* is in the direction of the wave vector.

Can we write this in a way that a human can understand?

Let's do this for the x-direction only case:

Wave phase speed: $c_p = \omega/k$

Wave phase speed: $c_p = c_r + ic_i$ – real *and* imaginary parts

$\eta' = Re\{\tilde{\eta}e^{ik(x - c_r t - ic_i t)}\}$

$\eta' = Re\{\tilde{\eta}e^{kc_i t}e^{ik(x - c_r t)}\}$

Recall: $e^{ix} = cos(x) + isin(x)$ – Euler's formula

$\eta' = Re\{\tilde{\eta}e^{kc_i t}[cos(k(x - c_r t)) + isin(k(x - c_r t))]\}$

$\eta' = \tilde{\eta}e^{kc_i t}cos(k(x - c_r t))$

$$\boxed{\eta' = \left(\tilde{\eta}e^{kc_i t}\right) cos(k(x - c_r t))} \tag{6}$$

- Exponential growth rate of wave amplitude: $\boxed{\sigma = kc_i}$ (positive = growing, negative = decaying; units: $[s^{-1}]$)
- Initial amplitude of wave: $\tilde{\eta}$
- Note that the cos() function repeats every 2π radians – this occurs over distance $x = \frac{2\pi}{K}$, and hence this is the wavelength of the wave.

Thus the **phase speed, c, tells you everything about how a wave will evolve**:

1. **real part** (c_r) = wave **propagation** ($\omega = kc_r$)
2. **imaginary part** (c_i) = exponential **growth/decay** of wave amplitude ($\sigma = kc_i$).

5 Unbalanced flow: Gravity waves

VallisE p70

Waves: motion in the presence of a **restoring force**

Simplest model:

- single fluid layer
- flat bottom
- free upper surface

(**FIG 4.1 with flat bottom**)

Restoring force: gravity

Fluid thickness: $h(x, y, t) = H + \eta'(x, y, t)$

Horizontal velocity: $\vec{\mathbf{u}}(x, y, t) = \vec{\mathbf{u}}'(x, y, t)$ (zero mean flow)

5.1 Non-rotating waves

Linearized continuity equation:

$\frac{\partial(H+\eta')}{\partial t} + \vec{\mathbf{u}}' \cdot \nabla(H+\eta') + (H+\eta')\nabla \cdot \vec{\mathbf{u}}' = 0$

$\frac{\partial \eta'}{\partial t} + \vec{\mathbf{u}}' \cdot \nabla \eta' + (H+\eta')\nabla \cdot \vec{\mathbf{u}}' = 0$

Linearization: neglect products of perturbation terms

$$\frac{\partial \eta'}{\partial t} + H\nabla \cdot \vec{\mathbf{u}}' = 0 \tag{7}$$

Similarly, linearized momentum equation:

$$\frac{\partial \vec{\mathbf{u}}'}{\partial t} = -g\nabla \eta' \tag{8}$$

Eliminate velocity:

d/dt of linearized continuity: $\frac{\partial^2 \eta'}{\partial t^2} + H\nabla \cdot \frac{\partial \vec{\mathbf{u}}'}{\partial t} = 0$

div of linearized momentum: $\nabla \cdot \frac{\partial \vec{\mathbf{u}}'}{\partial t} = -g\nabla^2 \eta'$, 2D horizontal Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

combine:

$$\frac{\partial^2 \eta'}{\partial t^2} - gH\nabla^2 \eta' = 0 \tag{9}$$

This is a **wave equation** (see VallisE Ch 6.0-6.2 for review of wave basics). Since the coefficients are constant, the solution will take the form:

$$\eta' = Re\{\tilde{\eta} e^{i(\hat{\mathbf{k}} \cdot \hat{\mathbf{x}} - \omega t)}\} \tag{10}$$

horizontal wavenumber: $\hat{\mathbf{k}} = \hat{\mathbf{i}}k + \hat{\mathbf{j}}l$

$\tilde{\eta}$ is a complex constant

Real part: Re

Simple application: 1D in x direction (constant in y)

$\eta' = Re\{\tilde{\eta} e^{i(kx-\omega t)}\}$

$\frac{\partial^2}{\partial t^2}(Re\{\tilde{\eta} e^{i(kx-\omega t)}\}) - gH\frac{\partial^2}{\partial x^2}(Re\{\tilde{\eta} e^{i(kx-\omega t)}\}) = 0$

$(-i\omega)^2 \eta' - gH(ik)^2 \eta' = 0$

$\omega^2 - gH(k^2) = 0$

Yields the dispersion relation relating wave frequency to wave number:

$$\omega = \pm\sqrt{gH}k \tag{11}$$

with gravity wave phase speed

$$c = \left|\frac{\omega}{k}\right| = \sqrt{gH} \tag{12}$$

The waves are **non-dispersive**: c is independent of k (thus all waves move at the same speed)

Since all waves move together and the general solution is a superposition of all waves, the general solution is:

$$\eta'(x,t) = \frac{1}{2}(F(x-ct) + F(x+ct)) \tag{13}$$

where $F(x)$ is the initial height field ($t = 0$).
Thus, an initial disturbance

- propagates both to the right and to the left at speed c
- preserves its initial shape

Analog: small waves in a bathtub simply bouncing around.

5.2 Rotating gravity waves (Poincare)

f-plane
Linearized equations:

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial \eta'}{\partial x} \tag{14}$$

$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial \eta'}{\partial y} \tag{15}$$

$$\frac{\partial \eta'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{16}$$

Try the solution

$$(u', v', \eta') = (\tilde{u}, \tilde{v}, \tilde{\eta})e^{i(\hat{\mathbf{k}}\cdot\hat{\mathbf{x}} - \omega t)} \tag{17}$$

Substitution yields set of homogeneous equations (matrix format):

$$\begin{pmatrix} -i\omega & -f_0 & igk \\ f_0 & -i\omega & igl \\ iHk & iHl & -i\omega \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} = 0$$

Non-trivial solutions only if determinant of matrix vanishes: (sum of (product of down diagonals) minus sum of (product of up diagonals))

$$\omega(\omega^2 - f_0^2 - c^2K^2) = 0 \tag{18}$$

Total horizontal wavenumber: $K^2 = k^2 + l^2$
Gravity wave phase speed: $c = \sqrt{gH}$
Two classes of solutions:

1. $\omega = 0$: geostrophic balance – time-independent flow (i.e. $\frac{\partial u'}{\partial t} = \frac{\partial v'}{\partial t} = \frac{\partial \eta'}{\partial t} = 0$), no waves (note: f-plane geostrophic wind is non-divergent, so this satisfies continuity equation as well; for varying f, this is not true → Rossby waves)
2. $\omega^2 = f_0^2 + c^2K^2$: Poincare waves – (Dispersion relation **FIG 4.4**), note: $\omega > f_0$.

Two limits of Poincare waves:

1. Short wave limit: $K^2 \gg \frac{f_0^2}{gH} \rightarrow \omega \approx \pm ck$ – same as non-rotating case (drop coriolis term from momentum eqn)
2. Long wave limit: $K^2 \ll \frac{f_0^2}{gH} \rightarrow \omega \approx \pm f_0$ – inertial oscillations (drop pressure gradient term from momentum eqn)

What does this mean? In terms of size of wave (i.e. wavelength λ)
"Short wave": $K = \frac{2\pi}{\lambda} \rightarrow$ (ignore 2π) $\lambda^2 \ll \frac{gH}{f_0^2} \rightarrow \lambda \ll L_d$
with **Rossby deformation radius**

$$L_d = \frac{\sqrt{gH}}{f_0} = \frac{c_{GW}}{f_0} \tag{19}$$

L_d **is the distance a shallow water gravity wave travels outward before being turned appreciably by the Coriolis acceleration. This limits how far away gravity waves can travel from their source!**

- "Short wave": $\lambda \ll L_d$ – do not feel rotation, act like non-rotating gravity waves
- "Long wave": $\lambda \gg L_d$ – do not propagate like gravity waves, simply turned by Coriolis

Key outcome: without rotation, gravity waves of all wavelengths respond the same; with rotation, this is no longer true!
You will explore a similar problem in the next Tank Lab: geostrophic adjustment.

5.3 Kelvin waves

A rotating gravity wave + a lateral solid boundary
(**FIG 4.5**)
Solid boundary = zero normal flow. Do wave-like (i.e. harmonic) solutions still exist?
Same linearized rotating SW equations:

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial \eta'}{\partial x} \tag{20}$$

$$\frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial \eta'}{\partial y} \tag{21}$$

$$\frac{\partial \eta'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{22}$$

Consider half-plane where $y > 0$ with boundary at $y = 0 \rightarrow v'(y = 0) = 0$. Try simple solution where $v' = 0$ everywhere.

$$\frac{\partial u'}{\partial t} = -g\frac{\partial \eta'}{\partial x} \tag{23}$$

$$f_0 u' = -g\frac{\partial \eta'}{\partial y} \tag{24}$$

$$\frac{\partial \eta'}{\partial t} + H\frac{\partial u'}{\partial x} = 0 \tag{25}$$

d/dt of u' equation: $\frac{\partial}{\partial t}\frac{\partial u'}{\partial t} = -g\frac{\partial}{\partial t}\frac{\partial \eta'}{\partial x}$
d/dx of η' equation: $\frac{\partial}{\partial x}\frac{\partial \eta'}{\partial t} + H\frac{\partial^2 u'}{\partial x^2} = 0$
Eliminate η':

$$\frac{\partial^2 u'}{\partial t^2} = c^2\frac{\partial^2 u'}{\partial x^2} \tag{26}$$

with usual gravity wave speed $c = \sqrt{gH}$.
The solution to this equation is:

$$u' = F_1(x+ct, y) + F_2(x-ct, y) \tag{27}$$

From x-momentum equation, the corresponding surface displacement η' in the x-direction is
$c(F_1(x+ct, y) - F_2(x-ct, y)) = -g\frac{\partial \eta'}{\partial x}$
$\frac{\partial \eta'}{\partial x} = -\frac{c}{g}(F_1(x+ct, y) - F_2(x-ct, y))$
$\frac{\partial \eta'}{\partial x} = -\sqrt{\frac{H}{g}}(F_1(x+ct, y) - F_2(x-ct, y))$
Integrate once in x (nothing changes):

$$\eta' = -\sqrt{\frac{H}{g}}(F_1(x+ct, y) - F_2(x-ct, y)) \tag{28}$$

This is a superposition of two waves, one traveling in the negative x direction and the other in the positive x direction – i.e. a basic non-rotating 1D gravity wave.

From y-momentum equation, the solution in the y-direction is:
$f_0(F_1 + F_2) = -g(-\sqrt{\frac{H}{g}}(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial y}))$
$f_0(F_1 + F_2) = \sqrt{gH}(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial y})$
Linear superposition, thus:
$\frac{\partial F_1}{\partial y} = \frac{f_0}{\sqrt{gH}}F_1$
$\frac{\partial F_2}{\partial y} = -\frac{f_0}{\sqrt{gH}}F_2$
The solutions for these equations are:

$$F_1 = F(x+ct)e^{\frac{y}{L_d}} \tag{29}$$

$$F_2 = G(x-ct)e^{\frac{-y}{L_d}} \tag{30}$$

with Rossby deformation radius $L_d = \frac{\sqrt{gH}}{f_0}$

For $y > 0$, F_1 blows up moving away from the boundary towards infinity and so is unphysical. Thus we keep F_2, which gives the solution

$$u' = e^{\frac{-y}{L_d}} G(x - ct) \tag{31}$$

$$v' = 0 \tag{32}$$

$$\eta' = \sqrt{\frac{H}{g}} e^{\frac{-y}{L_d}} G(x - ct) \tag{33}$$

These are Kelvin waves.

- decay exponentially away from the boundary
- boundary to the right for positive f_0 – turned to the right by the Coriolis

In words: these are regular SW gravity waves that are being constantly turned into the lateral boundary by the Coriolis force.

Example: in the NH, Kelvin waves turn clockwise around a barrier (e.g. a mountain range).

Equatorial Kelvin waves Kelvin waves are also found in the deep tropics, propagating along the equator – eastward in both hemispheres.

The equator also behaves *dynamically* like a barrier. Why? Imagine moving eastward just north of the equator in the northern hemisphere – you will be turned to the right by the Coriolis force, towards the equator. What happens if you cross the equator? How will the Coriolis force turn you?