1 Shallow water system: basics

Shallow water equations describe:

- a thin layer of fluid (\( H \ll L \))
- constant density (\( \rho_0 \))
- hydrostatic balance
- lower boundary = rigid surface
- upper boundary = free surface, above which is another fluid of negligible inertia

Benefits:

- Very useful model of effects of rotation on fluid without the complicating effects of stratification.
- Fluid motion fully determined by momentum + mass continuity equations (constant density = no thermodynamic variation in fluid)

Note: stacking multiple such layers of immiscible fluid of different densities = stably-stratified "stacked shallow water" system. This behaves like a continuously stratified system.

2 Equations

FIG 4.1
Fluid thickness: \( h(x, y, t) = \eta(x, y, t) - \eta_b(x, y, t) \)
Height of bottom surface (i.e. topography): \( \eta_b \), where \( \eta_b = 0 \)
Height of free surface: \( \eta \) (if flat bottomed, i.e. \( \eta_b = 0 \), then \( \eta = h \))

1) Momentum:
Vertical: hydrostatic balance
\[ \frac{\partial p}{\partial z} = -\rho_0 g \]
Integrate downwards from fluid surface, assume \( p(x, y, \eta, t) = 0 \) at top of fluid (i.e. assume overlying mass is negligible)
\[ \int_0^\eta \partial_p = -\rho_0 g \int_\eta^z \partial z \]
\[ 0 - p = -\rho_0 g(\eta(x, y, t) - z) \]
\[ p(x, y, t) = \rho_0 g(\eta(x, y, t) - z) \]

Horizontal:
Horizontal velocity: \( \mathbf{u} = (u, v) \)
The horizontal pressure gradient is independent of height!
\[ \nabla p = \rho_0 g \nabla \eta \]
\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f \mathbf{k} \times \mathbf{u} = -g \nabla \eta \quad [ms^{-2}] \tag{1}
\]
Note: the horizontal velocity \( \mathbf{u}(x, y) \) is also independent of height!

2) Mass (continuity):
Incompressible: \( \nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \)
\[
\frac{\partial w}{\partial z} = - \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\nabla \cdot \mathbf{u}
\]
Integrate upwards from the bottom to the top; \( \nabla \cdot \mathbf{u} \) is independent of height
\[ w(\eta) - w(\eta_b) = -h(\nabla \cdot \mathbf{u}) \]
\[ w(\eta) = \frac{Dh}{Dt} - \text{vertical velocity at fluid top literally changes the height of the fluid top} \]
\[ w(\eta_b) = \frac{Dh}{Dt} = 0 - \text{same for fluid bottom (e.g. an earthquake)} \]
Thus
\[ \frac{D}{Dt}(\eta - \eta_b) = -h \nabla \cdot \mathbf{u} \]
\[ \frac{Dh}{Dt} = -h \nabla \cdot \mathbf{u} \] - moving with the flow, the height will increase if the volume increases (flow convergence).
\[ \frac{Dh}{Dt} + h \nabla \cdot \mathbf{u} = 0 \]
\[
\frac{\partial h}{\partial t} + \mathbf{u} \cdot \nabla h + h \nabla \cdot \mathbf{u} = 0 \quad [ms^{-1}] \tag{2}
\]
Equivalently \( \frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0 \) - the height of the fluid increases if there is a convergent flux of height (!) by the flow.
Note: FIG 4.2 – VallisE gives an alternative derivation

3 Balanced flow: Geostrophic balance (VallisE Ch 3.3)
Geostrophic balance:
Scale analysis of LHS of horizontal momentum equation: \( \mathbf{u} \cdot \nabla \mathbf{u} \sim \frac{U^2}{L} \), \( f \mathbf{k} \times \mathbf{u} \sim fU \)
Ratio: Rossby number \( Ro = \frac{U}{fL} \)
If \( Ro \ll 1 \): Coriolis \( \gg \) advection – can neglect advection term
\[
\frac{\partial \mathbf{u}}{\partial t} + f \mathbf{k} \times \mathbf{u} = -g \nabla \eta \tag{3}
\]
Geostrophic balance:
\[ f \mathbf{k} \times \mathbf{u} = -g \nabla \eta \tag{4} \]
The geostrophic wind speed $|\mathbf{u}_g|$ is proportional to the slope of the surface (FIG 3.6). This is also true on pressure coordinates (height = height of pressure surface).

Figure 1: VallisE Fig 4.6: Geostrophic flow in one-layer shallow water system ($f > 0$).

**Fig. 4.6:**
Optional: Margules two-layer shallow water slope, thermal wind balance (cf. separate PDF notes)

4 Review: waves

VallisE 6.1-6.2. See box ”Wave Fundamentals” on p107, as well as Figure 2 below.

4.1 Wave basics: (VallisE Fig 6.1)

- **wave:** an oscillating motion, caused by a restoring force. Can move at some speed in a direction.

- **(angular) frequency:** $\omega$; the angular displacement per unit time (rate of change of phase of wave) at a fixed location.

- **(angular) wavenumber:** $(x: k; y: l; z: m)$; the number of radians per unit distance. A true vector ($\mathbf{k} = (k, l, m)$).
  
  - **wavevector:** $\mathbf{k} = (k, l, m)$; the direction of propagation and its magnitude.
  
  - **total wavenumber:** $K^2 = k^2 + l^2 + m^2$; magnitude of the wavevector

- **(linear) wavelength:** $\lambda = \frac{2\pi}{K}$; the linear distance between two consecutive wave peaks (or troughs, or any other phase of the wave). The inverse of wavenumber. $x$: $\lambda_x = \frac{2\pi}{k}$; $y$: $\lambda_y = \frac{2\pi}{l}$; $z$: $\lambda_z = \frac{2\pi}{m}$ — but NOT a true vector (there is no “wavelength vector” $\lambda$)!

Frequent point of confusion: we use angular wavenumber ($k$) in the standard equation to define a wave (see below). However, we typically think about the size of waves in terms of the spatial wavelength ($\lambda$). This is why we need the $2\pi$ factor when converting between the two — we care about the direct spatial distance from crest to crest, not the angular distance along the wave itself.
4.2 Wave motion basics: (VallisE Fig 6.2)

- **Dispersion relation**: $\omega(k)$; equation relating wave frequency and size (defined by the wavevector).

- **Phase speed**: $c_p = \omega/K$, $K = 2\pi/\lambda$; speed at which the wave crests move. $c_{px} = \omega/k$, $c_{py} = \omega/l$, $c_{pz} = \omega/m$ – but NOT a true vector (there is no “phase velocity” $\vec{c}_p$)!

- **Group velocity**: $\vec{c}_g = (c_{gx}, c_{gy}, c_{gz}) = \nabla_k \omega = (\partial \omega/\partial k, \partial \omega/\partial l, \partial \omega/\partial m)$; velocity at which a wave packet (i.e. envelope of waves and wave energy) moves. Most physical quantities of interest are transported at the group velocity. See VallisE 6.2.1 and Fig 6.2 to understand why the group velocity is defined as a derivative.

Notes:

- Standard convection is to define frequency ($\omega$) to be non-negative, i.e. $\omega \geq 0$. Wavenumber ($k$, $l$, $m$) may be negative or positive; positive = eastward/northward/upward; negative = westward/southward/downward.

- Why? The wavevector is $k \cdot x$, where $x$ is defined positive eastward/northward/upward. Thus $k \cdot x < 0$ means that e.g. for zonal motion $k < 0$ points opposite to the positive $x$ direction (eastward) and hence the wave will move westward.

4.3 Standard equation for monochromatic plane waves

$$\eta' = Re\{\tilde{\eta}e^ {i(kx - \omega t)}\}$$
Note: this can also be written as $\eta' = \text{Re}\{\tilde{\eta}e^{i(Kx^*-\omega t)}\}$, where $x^*$ is in the direction of the wave vector.

**Can we write this in a way that a human can understand?**

Let’s do this for the x-direction only case:

Wave phase speed: $c_p = \omega/k$
Wave phase speed: $c_p = c_r + ic_i$ – real and imaginary parts

\[
\eta' = \text{Re}\{\tilde{\eta}e^{ik(x-c_r t-ic_i t)}\}
\]
\[
\eta' = \text{Re}\{\tilde{\eta}e^{kct}e^{ik(x-c_r t)}\}
\]
Recall: $e^{ix} = \cos(x) + isin(x)$ – Euler’s formula

\[
\eta' = \text{Re}\{\tilde{\eta}e^{kct}[\cos(k(x-c_r t)) + isin(k(x-c_r t))]\}
\]
\[
\eta' = \tilde{\eta}e^{kct}\cos(k(x-c_r t))
\]

- Exponential growth rate of wave amplitude: $\sigma = kc_i$ (positive = growing, negative = decaying; units: $[s^{-1}]$)
- Initial amplitude of wave: $\tilde{\eta}$
- Note that the cos() function repeats every $2\pi$ radians – this occurs over distance $x = \frac{2\pi}{K}$, and hence this is the wavelength of the wave.

Thus the **phase speed, c, tells you everything about how a wave will evolve:**

1. **real part** ($c_r$) = wave propagation ($\omega = kc_r$)
2. **imaginary part** ($c_i$) = exponential growth/decay of wave amplitude ($\sigma = kc_i$).

## 5 Unbalanced flow: Gravity waves

VallisE p70

**Waves:** motion in the presence of a **restoring force**

Simplest model:

- single fluid layer
- flat bottom
- free upper surface

(FIG 4.1 with flat bottom)

Restoring force: gravity

Fluid thickness: $h(x,y,t) = H + \eta'(x,y,t)$
Horizontal velocity: $\vec{u}(x,y,t) = \vec{u}'(x,y,t)$ (zero mean flow)
5.1 Non-rotating waves

Linearized continuity equation:
\[
\frac{\partial (H + \eta^\prime)}{\partial t} + \mathbf{u}^\prime \cdot \nabla (H + \eta^\prime) + (H + \eta^\prime) \nabla \cdot \mathbf{u}^\prime = 0 \\
\frac{\partial \eta^\prime}{\partial t} + \mathbf{u}^\prime \cdot \nabla \eta^\prime + (H + \eta^\prime) \nabla \cdot \mathbf{u}^\prime = 0
\]
Linearization: neglect products of perturbation terms

\[
\frac{\partial \eta^\prime}{\partial t} + H \nabla \cdot \mathbf{u}^\prime = 0 \tag{7}
\]
Similarly, linearized momentum equation:

\[
\frac{\partial \mathbf{u}^\prime}{\partial t} = -g \nabla \eta^\prime \tag{8}
\]
Eliminate velocity:
d/dt of linearized continuity: \( \frac{\partial^2 \eta^\prime}{\partial t^2} + H \nabla \cdot \frac{\partial \mathbf{u}^\prime}{\partial t} = 0 \)
div of linearized momentum: \( \nabla \cdot \frac{\partial \mathbf{u}^\prime}{\partial t} = -g \nabla^2 \eta^\prime \), 2D horizontal Laplacian \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \)
combine:

\[
\frac{\partial^2 \eta^\prime}{\partial t^2} - gH \nabla^2 \eta^\prime = 0 \tag{9}
\]
This is a wave equation (see VallisE Ch 6.0-6.2 for review of wave basics). Since the coefficients are constant, the solution will take the form:

\[
\eta^\prime = \text{Re}\{\tilde{\eta} e^{i(kx - \omega t)}\} \tag{10}
\]
horizontal wavenumber: \( \hat{k} = \hat{i}k + \hat{j}l \)
\( \tilde{\eta} \) is a complex constant
Real part: \( \text{Re} \)
Simple application: 1D in x direction (constant in y)

\[
\eta^\prime = \text{Re}\{\tilde{\eta} e^{i(kx - \omega t)}\} \\
\frac{\partial^2}{\partial t^2} (\text{Re}\{\tilde{\eta} e^{i(kx - \omega t)}\}) - gH \frac{\partial^2}{\partial x^2} (\text{Re}\{\tilde{\eta} e^{i(kx - \omega t)}\}) = 0 \\
(-i\omega)^2 \eta^\prime - gH (ik)^2 \eta^\prime = 0 \\
\omega^2 - gH(k^2) = 0
\]
Yields the dispersion relation relating wave frequency to wave number:

\[
\omega = \pm \sqrt{gHk} \tag{11}
\]
with gravity wave phase speed

\[
c = \frac{|\omega|}{k} = \sqrt{gH} \tag{12}
\]
The waves are non-dispersive: \( c \) is independent of \( k \) (thus all waves move at the same speed)
Since all waves move together and the general solution is a superposition of all waves, the general solution is:
\[ \eta'(x, t) = \frac{1}{2}(F(x - ct) + F(x + ct)) \]  
\[ \text{where } F(x) \text{ is the initial height field } (t = 0). \]

Thus, an initial disturbance

- propagates both to the right and to the left at speed \( c \)
- preserves its initial shape

Analog: small waves in a bathtub simply bouncing around.

### 5.2 Rotating gravity waves (Poincare)

**f-plane**

Linearized equations:

\[
\begin{align*}
\frac{\partial u'}{\partial t} - f_0 v' &= -g \frac{\partial \eta'}{\partial x} \\
\frac{\partial v'}{\partial t} + f_0 u' &= -g \frac{\partial \eta'}{\partial y} \\
\frac{\partial \eta'}{\partial t} + H \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) &= 0
\end{align*}
\]

Try the solution

\[ (u', v', \eta') = (\tilde{u}, \tilde{v}, \tilde{\eta}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \]

Substitution yields set of homogeneous equations (matrix format):

\[
\begin{pmatrix}
-i\omega & -f_0 & igk \\
f_0 & -i\omega & igl \\
iHk & iHl & -i\omega
\end{pmatrix}
\begin{pmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{\eta}
\end{pmatrix} = 0
\]

Non-trivial solutions only if determinant of matrix vanishes: (sum of (product of down diagonals) minus sum of (product of up diagonals))

\[ \omega(\omega^2 - f_0^2 - c^2 K^2) = 0 \]

Total horizontal wavenumber: \( K^2 = k^2 + l^2 \)

Gravity wave phase speed: \( c = \sqrt{gH} \)

Two classes of solutions:

1. \( \omega = 0 \): geostrophic balance – time-independent flow (i.e. \( \frac{\partial u'}{\partial x} = \frac{\partial v'}{\partial x} = \frac{\partial \eta'}{\partial x} = 0 \)), no waves (note: f-plane geostrophic wind is non-divergent, so this satisfies continuity equation as well; for varying f, this is not true \( \rightarrow \) Rossby waves)

2. \( \omega^2 = f_0^2 + c^2 K^2 \): Poincare waves – (Dispersion relation FIG 4.4), note: \( \omega > f_0 \).
Two limits of Poincare waves:

1. Short wave limit: \( K^2 \gg \frac{f_0^2}{gH} \rightarrow \omega \approx \pm ck \) – same as non-rotating case (drop coriolis term from momentum eqn)

2. Long wave limit: \( K^2 \ll \frac{f_0^2}{gH} \rightarrow \omega \approx \pm f_0 \) – inertial oscillations (drop pressure gradient term from momentum eqn)

What does this mean? In terms of size of wave (i.e. wavelength \( \lambda \))

"Short wave": \( K = \frac{2\pi}{\lambda} \rightarrow \) (ignore \( 2\pi \)) \( \lambda^2 \ll \frac{gH}{f_0} \rightarrow \lambda \ll L_d \)

with Rossby deformation radius

\[
L_d = \frac{\sqrt{gH}}{f_0} = \frac{c_{GW}}{f_0}
\]  

(19)

\( L_d \) is the distance a shallow water gravity wave travels outward before being turned appreciably by the Coriolis acceleration. This limits how far away gravity waves can travel from their source!

- "Short wave": \( \lambda \ll L_d \) – do not feel rotation, act like non-rotating gravity waves
- "Long wave": \( \lambda \gg L_d \) – do not propagate like gravity waves, simply turned by Coriolis

Key outcome: without rotation, gravity waves of all wavelengths respond the same; with rotation, this is no longer true!

You will explore a similar problem in the next Tank Lab: geostrophic adjustment.

5.3 Kelvin waves

A rotating gravity wave + a lateral solid boundary

(FIG 4.5)

Solid boundary = zero normal flow. Do wave-like (i.e. harmonic) solutions still exist?

Same linearized rotating SW equations:

\[
\frac{\partial u'}{\partial t} - f_0v' = -g \frac{\partial \eta'}{\partial x}
\]  

(20)

\[
\frac{\partial v'}{\partial t} + f_0u' = -g \frac{\partial \eta'}{\partial y}
\]  

(21)

\[
\frac{\partial \eta'}{\partial t} + H \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = 0
\]  

(22)

Consider half-plane where \( y > 0 \) with boundary at \( y = 0 \rightarrow v'(y = 0) = 0 \). Try simple solution where \( v' = 0 \) everywhere.
\[
\frac{\partial u'}{\partial t} = -g \frac{\partial \eta'}{\partial x} \tag{23}
\]
\[
f_0 u' = -g \frac{\partial \eta'}{\partial y} \tag{24}
\]
\[
\frac{\partial \eta'}{\partial t} + H \frac{\partial u'}{\partial x} = 0 \tag{25}
\]

\[
d/\text{dt of } u' \text{ equation: } \frac{\partial}{\partial t} \frac{\partial u'}{\partial t} = -g \frac{\partial}{\partial t} \frac{\partial \eta'}{\partial x}
\]
\[
d/\text{dx of } \eta' \text{ equation: } \frac{\partial}{\partial x} \frac{\partial \eta'}{\partial t} + H \frac{\partial^2 u'}{\partial x^2} = 0
\]

Eliminate \( \eta' \):
\[
\frac{\partial^2 u'}{\partial t^2} = c^2 \frac{\partial^2 u'}{\partial x^2} \tag{26}
\]

with usual gravity wave speed \( c = \sqrt{gH} \).

The solution to this equation is:
\[
u' = F_1(x + ct, y) + F_2(x - ct, y) \tag{27}
\]

From x-momentum equation, the corresponding surface displacement \( \eta' \) in the x-direction is
\[
c(F_1(x + ct, y) - F_2(x - ct, y)) = -g \frac{\partial \eta'}{\partial x}
\]
\[
\frac{\partial \eta'}{\partial x} = -\frac{c}{g}(F_1(x + ct, y) - F_2(x - ct, y))
\]
\[
\frac{\partial \eta'}{\partial x} = -\sqrt{\frac{H}{g}}(F_1(x + ct, y) - F_2(x - ct, y))
\]
Integrate once in x (nothing changes):
\[
\eta' = -\sqrt{\frac{H}{g}}(F_1(x + ct, y) - F_2(x - ct, y)) \tag{28}
\]

This is a superposition of two waves, one traveling in the negative x direction and the other in the positive x direction – i.e. a basic non-rotating 1D gravity wave.

From y-momentum equation, the solution in the y-direction is:
\[
f_0(F_1 + F_2) = -g(-\sqrt{\frac{H}{g}} \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial y})
\]
\[
f_0(F_1 + F_2) = \sqrt{gH} \left( \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial y} \right)
\]
Linear superposition, thus:
\[
\frac{\partial F_1}{\partial y} = \frac{f_0}{\sqrt{gH}} F_1
\]
\[
\frac{\partial F_2}{\partial y} = -\frac{f_0}{\sqrt{gH}} F_2
\]
The solutions for these equations are:
\[
F_1 = F(x + ct)e^{\frac{y}{Ld}} \tag{29}
\]
\[
F_2 = G(x - ct)e^{-\frac{y}{Ld}} \tag{30}
\]
with Rossby deformation radius $L_d = \frac{\sqrt{gH}}{f_0}$.

For $y > 0$, $F_1$ blows up moving away from the boundary towards infinity and so is unphysical. Thus we keep $F_2$, which gives the solution

$$u' = e^{-y/L_d} G(x - ct)$$

$$v' = 0$$

$$\eta' = \sqrt{\frac{H}{g}} e^{-y/L_d} G(x - ct)$$

These are Kelvin waves.

- decay exponentially away from the boundary
- boundary to the right for positive $f_0$ – turned to the right by the Coriolis

In words: these are regular SW gravity waves that are being constantly turned into the lateral boundary by the Coriolis force.

Example: in the NH, Kelvin waves turn clockwise around a barrier (e.g. a mountain range).

**Equatorial Kelvin waves** Kelvin waves are also found in the deep tropics, propagating along the equator – eastward in both hemispheres.

The equator also behaves *dynamically* like a barrier. Why? Imagine moving eastward just north of the equator in the northern hemisphere – you will be turned to the right by the Coriolis force, towards the equator. What happens if you cross the equator? How will the Coriolis force turn you?