EAPS 53600, Spring 2020
Lec 04b: Potential vorticity, shallow water (VallisE 4.2)

1 Recall: shallow water equations

Momentum:

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} + \vec{\mathbf{u}} \cdot \nabla \vec{\mathbf{u}} + f\hat{\mathbf{k}} \times \vec{\mathbf{u}} = -g\nabla\eta \quad [ms^{-2}] \tag{1}$$

Mass (continuity):

$$\frac{Dh}{Dt} + h\nabla \cdot \vec{\mathbf{u}} = 0 \tag{2}$$

2 Relative vorticity equation

Relative vorticity of fluid (3D):

$$\omega = \nabla \times \vec{\mathbf{v}} \tag{3}$$

Shallow water (2D): horizontal only

$$\zeta = \hat{\mathbf{k}} \cdot (\nabla \times \vec{\mathbf{v}}) = \nabla \times \vec{\mathbf{u}} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{4}$$

Take curl ($\nabla\times$) of this equation + vector identities (VallisE p69): **vorticity equation**

$$\frac{\partial \zeta}{\partial t} + \vec{\mathbf{u}} \cdot \nabla(\zeta + f) = -(\zeta + f)(\nabla \cdot \vec{\mathbf{u}}) \quad [s^{-2}] \tag{5}$$

f is constant in time, so can write:

$$\frac{D(\zeta + f)}{Dt} = -(\zeta + f)(\nabla \cdot \vec{\mathbf{u}}) \tag{6}$$

Note: **absolute vorticity** is $\eta = \zeta + f$ (f is **planetary vorticity**).

3 Potential vorticity (PV): shallow-water

Combines vorticity and depth of fluid into a single quantity!

Use mass continuity equation to eliminate $\nabla \cdot \vec{\mathbf{u}}$ in vorticity equation:

$$\frac{1}{\zeta + f}\frac{D(\zeta + f)}{Dt} = \frac{1}{h}\frac{Dh}{Dt} \tag{7}$$

$\frac{D(ln(\zeta+f))}{Dt} = \frac{D(ln(h))}{Dt}$
$\frac{D}{Dt}(ln(\zeta + f) - ln(h)) = 0$

$$\frac{DQ}{Dt} = 0 \tag{8}$$

Shallow-water potential vorticity (PV):

$$Q = \frac{\zeta + f}{h} \tag{9}$$

A material invariant (i.e. conserved moving with the flow).

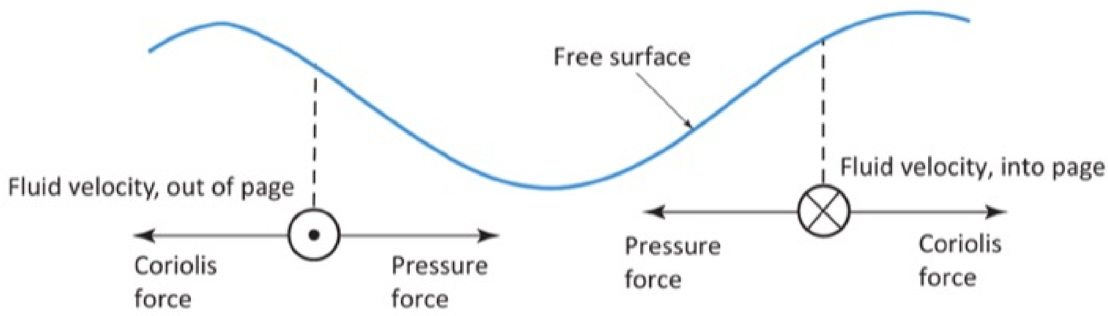

Fig. 4.6:

Figure 1: VallisE Fig 4.6: Geostrophic flow in one-layer shallow water system ($f > 0$).

Takeaway: the dynamics of the full shallow water system can be described by conservation of a single quantity – potential vorticity.

GROUP: In this figure, what is the sign of the vorticity of the column of fluid where the fluid is shallowest? Deepest? Where is the potential vorticity largest and smallest?

Note: why a "column" of fluid? Recall that in the shallow water system there is no vertical variation in flow speed (or density), so the column of fluid acts in unison; in other words, the only property representing the vertical dimension is the fluid depth h.

GROUP: Consider a column of fluid that conserves its potential vorticity as it moves.

- If the column becomes taller (i.e. "column stretching"), how would its absolute vorticity change?
- If the column is stretched and f is constant, how would its relative vorticity change?
- If the column moves poleward and h is constant, how would its relative vorticity change?
- If the column moves eastward down a mountain slope and its top remains at the same height (e.g. the tropopause), how would its relative vorticity change?

The rotating tank is an f-plane, so f cannot be (easily) varied within the tank. Could you think of a clever way to represent the *dynamical* effects of varying f in the tank?