EAPS 53600, Spring 2020
Lec 04c: Geostrophic adjustment

1 Intro

VallisE p74 (note: book uses a different type of example for this concept!)

If you impose a perturbation (bump) in a shallow layer of fluid, how does it evolve?

Top-hat IC: rotating slab of finite width $2L$ in x-direction and constant height H (**FIG DRC3**)

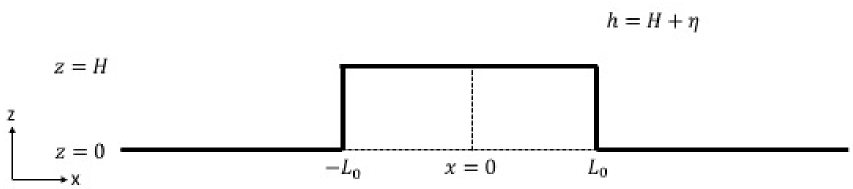

Fig. DRC3. 1D rotating top-hat perturbation

Initial state ($t = 0$):
Initial height:

$$h_0(x) = \begin{cases} H & \text{if } |x| < L \\ 0 & \text{if } |x| > L \end{cases}$$

Initial velocity: zero motion: $u_0(x) = v_0(x) = 0$
Notes:

- an unbalanced perturbation – height perturbations without winds (and Coriolis force) to balance pressure gradient force.
- fourier decomposition = superposition of many waves of different wavelengths

Analog: Tank lab #2!

2 Shallow water potential vorticity

Rotating shallow water potential vorticity:

$$Q = \frac{\zeta + f}{h} \tag{1}$$

with relative vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$
A material invariant (i.e. conserved):

$$\frac{DQ}{Dt} = 0 \tag{2}$$

3 Analytic solution

Perturbation $h(x) = H + \eta(x)$ (i.e. $\eta \leq 0$ in this problem)
What is the final shape of $h(x)$?
Final state ($t \to \infty$):

- geostrophic balance $fv_g = g\frac{\partial \eta}{\partial x}$
- u = 0 (zero outward motion)

How to solve?

- geostrophic balance
- PV conservation
- total mass conservation

PV (full):

- $t = 0$: $Q_0 = \frac{f}{H}$ (i.e. constant everywhere)
- $t \to \infty$: $Q_f = \frac{\zeta + f}{H + \eta}$

Geostrophic relative vorticity: $\zeta = \frac{\partial v_g}{\partial x} = \frac{g}{f}\frac{\partial^2 \eta}{\partial x^2}$
PV conservation (lagrangian): $Q_f = Q_0$ (fluid parcels in top hat conserve their PV as they move)

$$\frac{\frac{g}{f}\frac{\partial^2 \eta}{\partial x^2} + f}{H + \eta} = \frac{f}{H} \quad [m^{-1}s^{-1}] \tag{3}$$

$\frac{g}{f}\frac{\partial^2 \eta}{\partial x^2} + f = \frac{f}{H}(H + \eta) = f + \frac{f}{H}\eta$
$\frac{\partial^2 \eta}{\partial x^2} - \frac{f^2}{gH}\eta = 0$

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{L_d^2}\eta = 0 \quad [m^{-1}] \tag{4}$$

Rossby deformation radius:

$$L_d = \frac{\sqrt{gH}}{f} \quad [m] \tag{5}$$

2nd-order ODE: need 2 BCs
1) Symmetry about $x = 0$
Valid solution:

$$\eta = \eta_0 cosh\left(\frac{x}{L_d}\right) \quad [m] \tag{6}$$

$(cosh(x) = \frac{1}{2}(e^x + e^{-x}),\ sinh(x) = \frac{1}{2}(e^x - e^{-x}))$
2) Total mass conservation ($M_f = M_0$) to solve for η_0

$$\rho_0 \int_0^L (H + \eta)dx = \rho_0 H L_0 \quad [kg\, m^{-1}] \tag{7}$$

where L is some large radius
$\int_0^L (H + \eta_0 cosh\left(\frac{x}{L_d}\right))dx = HL_0$
$HL + \eta_0 L_d sinh\left(\frac{L}{L_d}\right) = HL_0$

$$\eta_0 = -H\frac{(L - L_0)}{L_d}\frac{1}{sinh\left(\frac{L}{L_d}\right)} \quad [m] \tag{8}$$

Plug into solution:

$$\eta(x) = -H\frac{(L - L_0)}{L_d} coth\left(\frac{x}{L_d}\right) \quad [m] \tag{9}$$

Suppose $\eta = -H$ at $x = L$ (i.e. outer edge of final perturbation, i.e. $h = 0$)

$$z - tanh(z) = z_0 \tag{10}$$

where

$$z = \frac{L}{L_d} \tag{11}$$

Note: $tanh(z)$ bounded from $[-1, 1]$ from $z = (-\infty, \infty)$.
Solution? Consider two limits
(**FIG DRC5**)
1) Case 1: $L_0 \gg L_d$ (i.e. $z_0 \gg 1$)
Then $z \gg 1$ too (because $|tanh(z)| \leq 1$)
Thus, $z \to z_0 + 1$ i.e.

$$L \to L_0 + L_d \qquad (L_0 \gg L_d) \tag{12}$$

Fluid spreads out by **one deformation radius** then stops. Since $L_0 \gg L_d$: fluid barely spreads out.

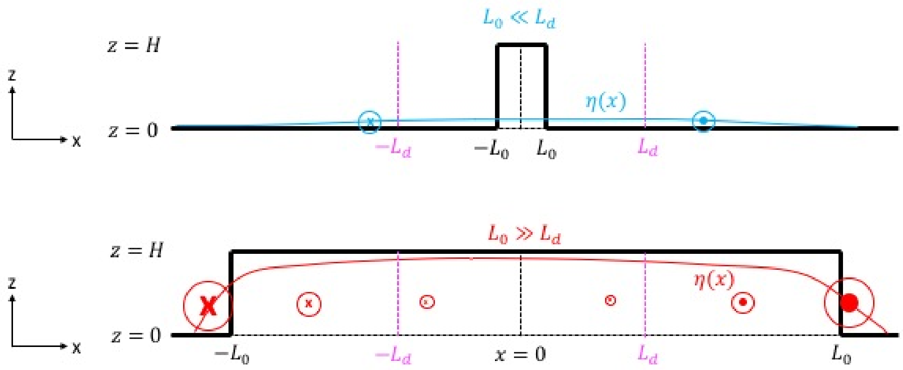

Fig. DRC5. Schematic of initial + final states for top-hat perturbation of different length scales (f>0)

Meanwhile, geostrophic wind response is strong: $v_g \frac{g}{f}\frac{\partial \eta}{\partial x}$ – for $f > 0$: on the negative-x side, $\frac{\partial \eta}{\partial x} > 0 \rightarrow v_g > 0$ (i.e. into page); on the positive-x side, $\frac{\partial \eta}{\partial x} < 0 \rightarrow v_g < 0$ (i.e. out of page).

2) Case 2: $L_0 \ll L_d$ (i.e. $z_0 \ll 1$)

$lim_{z \rightarrow 0} tanh(z) \approx z - \frac{1}{3}z^3 + \dots$

Thus, $\frac{1}{3}z^3 \rightarrow z_0$

$z \rightarrow (3z_0)^{\frac{1}{3}}$

$\frac{L}{L_d} \rightarrow (3\frac{L_0}{L_d})^{\frac{1}{3}}$

$$L \rightarrow 3^{\frac{1}{3}} L_0^{\frac{1}{3}} L_d^{\frac{2}{3}} \qquad (L_0 \ll L_d) \tag{13}$$

Geometric average of L_0 and L_d, weighted towards L_d

This time, geostrophic wind response is weak since $\frac{\partial \eta}{\partial x}$ is relatively small.

Full solution: **FIG DRC4**

Thus, two limits = two classes of systems

- $L \gg L_d$: strongly constrained by rotation – mass (i.e. height) field adjusts weakly, flow (i.e. velocity) field adjusts strongly to geostrophic flow ("quasi-balanced"). ("winds adjust to the mass")
- $L \ll L_d$: weakly constrained by rotation – mass (i.e. height) field adjusts strongly, flow (i.e. velocity) field adjusts weakly to geostrophic flow. ("mass adjusts to the wind")

What is $L_d = \frac{\sqrt{gH}}{f}$?

$L = \frac{U}{\tau}$

$U = \sqrt{gH} = c_{GW}$

$\tau_{Cor} = \frac{1}{f}$ – timescale of Coriolis acceleration (i.e. "feeling the Earth's rotation", $|a_{Cor}| = fU$)

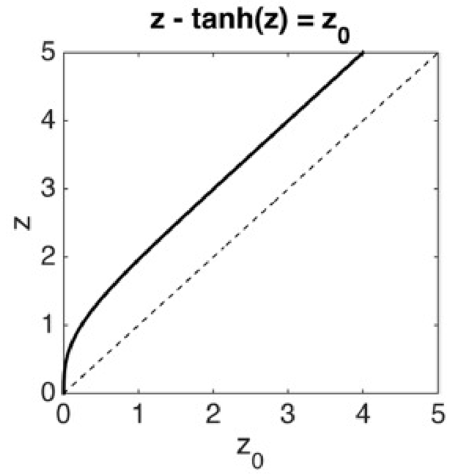

Fig. DRC4. z solution for final state 1D rotating top-hat perturbation

$(= \frac{1}{2\Omega sin\phi} = \frac{1}{2(\frac{2\pi}{1\ siderealday})sin\phi} = \frac{\frac{1}{4\pi}\ siderealday}{sin\phi}$ – proportional to a "pendulum day" $\frac{1\ siderealday}{sin(\phi)})$

Thus $L_d = c_{GW}\tau_{Cor}$ – L_d **is the distance a shallow water gravity wave travels outward before being turned appreciably by the Coriolis acceleration.**

Note: the non-rotating case is simply $f = 0$ (and thus $L_d = \infty$), i.e. no rotation, only gravity waves radiating out to ∞!

Note: what about conservation of energy? It turns out that $E_f < E_0$. Where does the extra energy go? (Homework.)

Example – waves in a tank of water

Rotating at 15 rpm, $H = 0.1\ m$, $g' \approx 0.05 * 10\ ms^{-2} = 0.5\ ms^{-2}$, $f_0 = 2\Omega = 2 * (\frac{15\ rev}{1\ min} * \frac{1\ min}{60\ s} * \frac{2\pi\ rad}{1rev}) = \pi\ s^{-1}$ gives

$c_{GW} = \sqrt{gH} = 0.22\ ms^{-1}$ (recall the tank is small)

$L_d = \frac{c_{GW}}{f_0} = \frac{0.22\ ms^{-1}}{pi\ s^{-1}} = 0.07\ m = 7\ cm$

How does this compare with L_0 in the tank experiment?

Example – atmosphere

What is L_d at the equator? ∞

What is L_d at the poles? Take $H = 10\ km$, $g \approx 10\ ms^{-2}$, $f = 1.46 * 10^{-4}\ s^{-1}$

$c_{GW} = \sqrt{gH} = 316\ ms^{-1}$ $L_d = \frac{316\ ms^{-1}}{1.46*10^{-4}\ s^{-1}} \approx 2000\ km$

Here at Purdue: $L_d \approx 3000\ km$ – length scale of pair of extratropical cyclone + anti-cyclone (think one United States)

4 Why are we talking about this?

POWERPOINT Maps

Extratropics: larger f = smaller L_d – extratropical cyclones, significant pressure gradients can exist in balance with Coriolis acceleration

Tropics: small f = large L_d – significant pressure gradients cannot be balanced with Coriolis acceleration (”weak temperature gradient approximation”)