EAPS 53600, Spring 2020
Margules derivation (thermal wind balance for two-layer shallow water system)

For a single-layer shallow water system, geostrophic balance is given by.

$$f\hat{\mathbf{k}} \times \vec{\mathbf{u}}_g = -g\nabla\eta \tag{1}$$

The geostrophic wind speed $|\vec{\mathbf{u}}_g|$ is proportional to the slope of the surface (FIG 3.6). This is also true on pressure coordinates (height = height of pressure surface).

Now, consider a two-layer shallow water system separated by an interface at height η (Figure 1 below). This is an extension of the single-layer system, in which we assumed the overlying fluid had negligible inertia (i.e. $\rho_1 \ll \rho_2$), to a system where the density of the overlying fluid is not negligible.

The following is based on the diagram in Figure 1. Subscript 1 refers to the upper (less dense) layer and subscript 2 refers to the lower (more dense) layer.

$\Delta p = p_1 - p_2 = p_1 - (p_1 + \rho_1 g(\Delta z - \eta) + \rho_2 g\eta)$

$\Delta p = -g((\rho_2 - \rho_1)\eta + \rho_1\Delta z)$

$-\frac{1}{\rho_1}\nabla(\Delta p) = g'\nabla\eta$

where $g' = g\left(\frac{\rho_2-\rho_1}{\rho_1}\right)$ is called the reduced gravity.

Next, consider the geostrophic wind within each layer:

$\vec{\mathbf{u}}_i = \frac{1}{f\rho_i}\hat{\mathbf{k}} \times \nabla p_i$ for each layer i

Difference in wind vectors top - bottom (i.e. wind shear): $\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2 = \frac{1}{f}\left(\frac{1}{\rho_1}\hat{\mathbf{k}} \times \nabla p_1 - \frac{1}{\rho_2}\hat{\mathbf{k}} \times \nabla p_2\right)$

Boussinesq approximation ($\rho_2 \approx \rho_1$):

$\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2 = \frac{1}{f}\hat{\mathbf{k}} \times (\frac{1}{\rho_1}\nabla(\Delta p))$

Substituting from above yields:

$$\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2 = -\frac{g'}{f}\hat{\mathbf{k}} \times (\nabla\eta) \tag{2}$$

This equation is thermal wind balance for our two-layer shallow water system: vertical wind shear is proportional to the interface (i.e. frontal) slope (perpendicular to the shear vector).

1D version (Margules relation):

$$u_1 - u_2 = \frac{g'}{f}\frac{\partial\eta}{\partial y} \tag{3}$$

upward meridional slope = positive zonal wind shear (i.e. a jet)

Link to Earth's atmosphere: meridionally-decreasing temperature ($\frac{\partial T}{\partial y} < 0$) = meridionally-increasing density ($\frac{\partial\rho}{\partial y} > 0$) = upward sloping interface towards the pole ($\frac{\partial\eta}{\partial y} > 0$).

Warning: it may be tempting to set $u_2 = 0$ (e.g. surface friction), which yields an equation that *looks* like geostrophic balance, but beware! It has g' and not g (and the wrong sign for geostrophic balance – coriolis and pressure gradient accelerations both point in the same direction)

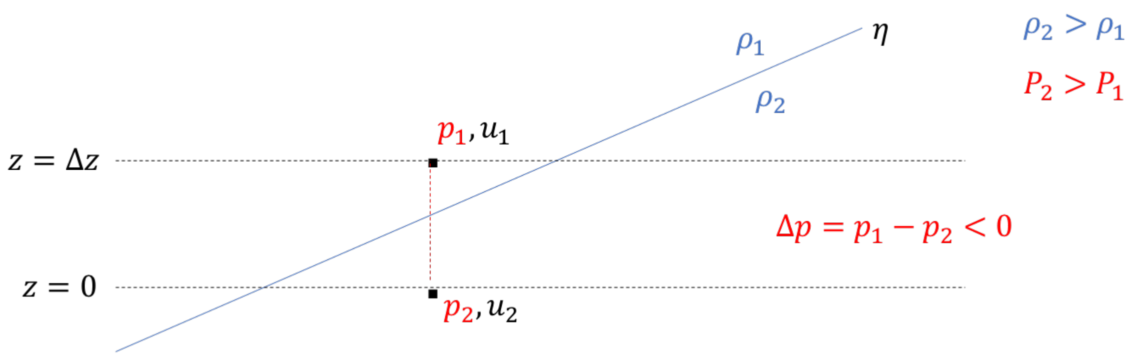

Figure 1: Two-layer shallow water system separated by sloping surface at height η.