EAPS 53600, Spring 2020
Lec 05: Quasi-geostrophic (QG) system

1 Intro

Quasi-geostrophic (QG) equations:

- Most widely used set of equations for atmos/ocean theory
- Want to understand horizontal scales of motion similar to the deformation radius (larger scales: pure geostrophic)

Motivation:

- Large-scale flow in the atmopshere/ocean is *close* to geostrophic and hydrostatic balance, horizontal scales similar to deformation radius
- Want a filtered set of equations appropriate to these specific scales of motion
- The equations only describe the motion on these scales, do **not** explain their existence

Assumptions:

1. Flow is near-geostrophic balance: small Rossby number, $Ro = \frac{U}{fL} \ll 1$
2. Small variations in f: $f = f_0 + \beta y$ (beta-plane), $\beta y \ll f_0$
3. Horizontal scale of motion similar to deformation radius: $L \sim L_d$

2 Shallow-water quasi-geostrophic (QG) potential vorticity (VallisE 5.2.2-5.2.3)

Outcomes of our assumptions:

- Small Ro: i) $\vec{\mathbf{u}} = \vec{\mathbf{u}_g} + \vec{\mathbf{u}_a}$, where $|\vec{\mathbf{u}_a}| \ll |\vec{\mathbf{u}_g}|$; ii) $\zeta \ll f_0$
- f nearly constant: $\nabla \cdot \mathbf{u}_g = 0$ (recall: for f constant, the geostrophic wind is non-divergent)

The QG system is simply a linearized version of the full system. Similarly, the system has a linearized version of the full PV called the quasi-geostrophic potential vorticity.

Note: for derivation from vorticity equation, see VallisE p87-88

Recall SW PV:

$$Q = \frac{f + \zeta}{h} \tag{1}$$

Material conservation: $\frac{DQ}{Dt} = 0$

Fluid depth: $h = H + h'$, where $h' = \eta_T - \eta_b$ is the deviation (top + bottom) from H

$h = H\left(1+\frac{h'}{H}\right)$
Beta plane: $f = f_0 + \beta y$
$Q = (f_0 + \beta y + \zeta)(\frac{1}{H})\frac{1}{1+\frac{h'}{H}}$
Linearize:

- Assume $\frac{h'}{H} \ll 1$ – taylor expansion: $Q \approx (f_0 + \beta y + \zeta)(\frac{1}{H})(1 - \frac{h'}{H})$
- neglect products of perturbations from H and f_0 (i.e. h', βy, ζ)
- $\zeta \approx \zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$

$Q \approx (\frac{1}{H})((f_0 + \beta y + \zeta_g) - \frac{f_0}{H}h')$
Divide through by $\frac{1}{H}$, and drop constant f_0 term (it's a constant so doesn't do anything).
Shallow water quasi-geostrophic potential vorticity:

$$\boxed{q = \beta y + \zeta_g - \frac{f_0}{H}h'} \tag{2}$$

Material conservation: advection only by the **geostrophic** wind, i.e.

$$\frac{D_g q}{Dt} = \frac{\partial q}{\partial t} + u_g \frac{\partial q}{\partial x} + v_g \frac{\partial q}{\partial y} = 0 \tag{3}$$

GROUP: Consider a column of fluid that conserves its potential vorticity as it moves.

- If the column becomes taller (i.e. "column stretching"), how would its absolute vorticity change?
- If the column is stretched and f is constant, how would its relative vorticity change?
- If the column moves poleward and h is constant, how would its relative vorticity change?
- If the column moves eastward down a mountain slope and its top remains at the same height (e.g. the tropopause), how would its relative vorticity change?
- Do your answers differ from when using the full PV formulation?

Note: the effect of fluid depth is *divided* for full PV but *subtracted* for QGPV – both yield reduced PV for larger h.

Summary: QGPV is a linearized approximation to full PV. (Note: the conservation equation itself is *not* linearized.)

2.1 Formulation in terms of streamfunction (ψ) and deformation radius (L_d)

The above formulation is nice, but h' and ζ are actually related to each other...

Because the geostrophic wind (for $f = f_0$ constant) is **non-divergent**, we can write the horizontal flow field in terms of a **streamfunction**, ψ:

$$u_g = -\frac{\partial \psi}{\partial y} \tag{4}$$

$$v_g = \frac{\partial \psi}{\partial x} \tag{5}$$

Non-divergent flow moves parallel to the steamfunction contours, like water flowing between the banks of a stream. (Is it really non-divergent? Check for yourself: try taking $\nabla \cdot \vec{\mathbf{u}}_g$)

In the shallow water system, the **geostrophic streamfunction** is:

$$\boxed{\psi = \frac{gh'}{f_0}} \tag{6}$$

and so

$$u_g = -\frac{g}{f_0}\frac{\partial h'}{\partial y} \tag{7}$$

$$v_g = \frac{g}{f_0}\frac{\partial h'}{\partial x} \tag{8}$$

The geostrophic relative vorticity is:

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{g}{f_0}\nabla^2 h' = \nabla^2 \psi \tag{9}$$

Finally, recall the deformation radius:

$$\boxed{L_d = \frac{c_{GW}}{f_0}} \tag{10}$$

where in the shallow water

$$\boxed{c_{GW} = \sqrt{gH}} \tag{11}$$

Using the above, we can rewrite the last term of the QG PV as:
$\frac{f_0}{H}h' = \frac{f_0}{H}\left(\frac{f_0\psi}{g}\right) = \frac{f_0^2}{gH}\psi = \frac{1}{L_d^2}\psi$
which yields an alternate form of the QG PV equation:

$$\boxed{q = \beta y + \nabla^2\psi - \frac{1}{L_d^2}\psi} \tag{12}$$

with material derivative

$$\frac{D_g q}{Dt} = \frac{\partial q}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} \tag{13}$$

and thus budget equation

$$\frac{D_g q}{Dt} = \frac{\partial q}{\partial t} - J(\psi, q) = 0 \tag{14}$$

Jacobian $J(A,B) = \frac{\partial A}{\partial y}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial y}$

Note: this formulation is conveniently written entirely in terms of ψ and q, and the only external parameters are now L_d and β!

This is very convenient, because it can be used directly to extend our shallow water system thinking to a continuously-stratified system (i.e. the real atmosphere).

3 Continuously-stratified system (VallisE 5.5)

In a continuously-stratified fluid, the gravity wave phase speed is slightly different:

$$\boxed{c_{GW} = NH} \tag{15}$$

where

$$N = \sqrt{\frac{-g}{\tilde{\rho}}\frac{\partial \tilde{\rho}}{\partial z}} \qquad [s^{-1}] \tag{16}$$

is the Brunt-Vaisala frequency (cf. VallisE p59) – a.k.a. the "**buoyancy frequency**", i.e. the frequency at which an air parcel would oscillate up and down in a stably-stratified environment. (think of a balloon bobbing slowly up and down). $\tilde{\rho}(z)$ is the background vertical variation in density.

Hence, the deformation radius is:

$$\boxed{L_d = \frac{c_{GW}}{f} = \frac{NH}{f}} \tag{17}$$

The geostrophic streamfunction is now:

$$\boxed{\psi = \frac{p'}{f_0 \rho_0}} \tag{18}$$

and so

$$u_g = -\frac{1}{\rho_0 f_0}\frac{\partial p}{\partial y} \tag{19}$$

$$v_g = \frac{1}{\rho_0 f_0}\frac{\partial p}{\partial x} \tag{20}$$

(look familiar? this is regular old geostrophic wind as taught in meteorology)

Now, since we're allowing density to vary, we need to include the **thermodynamic equation** too.

$$\frac{Db}{Dt} = 0 \tag{21}$$

where b is buoyancy (recall: positive buoyancy = less dense = (ideal gas) warmer).

Let's use the boussinesq approximation (i.e. background density constant, background buoyancy varies only with height) for simplicity:

$$b(x,y,z,t) = \tilde{b}(z) + b'(x,y,t) \tag{22}$$

plugging in yields

$$\frac{Db'}{Dt} + w\frac{\partial \tilde{b}}{\partial z} = 0 \tag{23}$$

The buoyancy frequency is itself defined as the background vertical gradient in buoyancy, i.e. $N^2 = \frac{\partial \tilde{b}}{\partial z}$. Thus,

$$\frac{Db'}{Dt} + N^2 w = 0 \tag{24}$$

Finally, we return to the original quasi-geostrophic vorticity equation:

$$\frac{D_g(\zeta + f)}{Dt} = -f_0(\nabla \cdot \vec{\mathbf{u}}) \tag{25}$$

Using 3D mass continuity ($\nabla \cdot \vec{\mathbf{v}} = 0$), we can rewrite this as

$$\frac{D_g(\zeta + f)}{Dt} = f_0(\frac{\partial w}{\partial z}) \tag{26}$$

The QG version of the thermodynamic equation is:

$$\frac{D_g b'}{Dt} + N^2 w = 0 \tag{27}$$

Combining the above two equations to eliminate w yields:

$$\frac{Dq_g}{Dt} = 0 \tag{28}$$

where

$$\boxed{q = \beta y + \zeta_g + \tfrac{\partial}{\partial z}\left(\tfrac{f_0}{N^2} b'\right)} \tag{29}$$

Amazingly, buoyancy is also related to the streamfunction via hydrostatic balance:

$$b' = f_0 \frac{\partial \psi}{\partial z} \tag{30}$$

(Why? $f_0 \frac{\partial \psi}{\partial z} = \frac{1}{\rho_0}\frac{\partial p'}{\partial z} = -\frac{1}{\rho_0}(\rho' g) = -g(\rho'/\rho_0) = b'$)

Hence, the alternative form for QG PV is:

$$\boxed{q = \beta y + \nabla^2 \psi - f_0^2 \tfrac{\partial}{\partial z}\left(\tfrac{1}{N^2}\tfrac{\partial \psi}{\partial z}\right)} \tag{31}$$

Notice that if N^2 is constant (i.e. the fluid stratification doesn't vary with height), you get

$$q = \beta y + \nabla^2 \psi - \frac{H^2}{L_d^2}\frac{\partial^2 \psi}{\partial z^2} \tag{32}$$

Compare this with the shallow-water version above. Note that the two partial derivatives in z act like a factor $1/H^2$.

Note: the continuously stratified system is conceptually identical to the shallow-water system; the math is just a bit more complicated.

- Pressure has replaced fluid height as the relevant dynamical quantity.
- However, recall that if one uses pressure coordinates instead of height, then pressure is replaced with geopotential height.
- This is directly analogous to the shallow water fluid height – higher heights = higher pressure, negative vorticity; lower heights = lower pressure, positive vorticity.