EAPS 53600, Spring 2020
Lec 06: Rossby Waves (VallisE 6.3-6.5)

1 Intro

See summary box "Essentials of Rossby Waves" VallisE p113.

- Perhaps most important large-scale wave in the atmos/ocn
- Simplest system: *linearized* QG system – we will now linearize the governing equation (i.e. the D_g/Dt operator; QG PV was already a linearized version of full PV)

$$\frac{\partial q}{\partial t} + \vec{\mathbf{u}} \cdot \nabla q = 0 \tag{1}$$

Linearize the equation: (more general)

- flow = time-independent component (**basic state**, i.e. a constant background state) + perturbation
- perturbation $\ll$ basic state
- here: **linearize about a basic state given by pure zonal flow $\overline{\vec{\mathbf{u}}}(y)$** (and so $\overline{v} = 0$)

$$q = \overline{q}(y) + q'(x, y, t) \tag{2}$$
$$\psi = \overline{\psi}(y) + \psi'(x, y, t) \tag{3}$$
$$u = \overline{u}(y) + u'(x, y, t) \tag{4}$$
$$v = v'(x, y, t) \tag{5}$$

Note: $\overline{u} = -\frac{\partial \overline{\psi}}{\partial y}$
Substitution: $\frac{\partial q'}{\partial t} + \frac{\partial \overline{q}}{\partial t} + \overline{\vec{\mathbf{u}}} \cdot \nabla \overline{q} + \overline{\vec{\mathbf{u}}} \cdot \nabla q' + \vec{\mathbf{u}}' \cdot \nabla \overline{q} + \vec{\mathbf{u}}' \cdot \nabla q' = 0$

- $\frac{\partial \overline{q}}{\partial t} = 0$
- basic state alone must satisfy governing equation, thus $\overline{\vec{\mathbf{u}}} \cdot \nabla \overline{q} = 0$
- neglect product of perturbations: $\vec{\mathbf{u}}' \cdot \nabla q' \approx 0$
- $\overline{v} = 0$, $\frac{\partial \overline{q}}{\partial x} = 0$

Result:

$$\boxed{\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0} \tag{6}$$

Notes:

- **Here we are looking at the behavior of the system associated *solely* with the governing equation for (QG) PV. Since this equation assumes that the flow is in (near) geostrophic balance, this equation *filters out unbalanced motions – i.e. gravity waves – from the system.*** The result will be solely Rossby waves!

- The next lecture generalizes the approach presented below to include *both* (unbalanced) gravity waves and (balanced) Rossby waves – this is the most general (and complicated) form of the equations on a "midlatitude" beta plane – i.e. where $|\beta y| \ll f_0$.

- The lecture after that will then address how things change for the "equatorial" beta plane – i.e. where $f_0 = 0$ and $f = \beta y$.

For now, though, let's get back to pure Rossby waves!

2 Basic state

Simplest option: constant zonal flow $\overline{u} = U$ constant

Recall $\overline{u} = -\frac{\partial \psi'}{\partial y}$. Thus $\overline{\psi} = -Uy \rightarrow \overline{h} = -(\frac{f_0}{g})Uy$.

Thus, the basic state of our single layer of fluid has:

- velocity: constant zonal flow

- mass: surface with gradual slope in y direction. For $U > 0$ (Earth: westerlies), $\overline{h}$ slopes *downwards* moving poleward (i.e. total fluid depth h decreases moving poleward). This is identical to how pressure surfaces slope downwards moving poleward in our atmosphere!

3 Flow with infinite L_d

Recall: $q = \beta y + \zeta_g - \frac{1}{L_d^2}\psi'$

If $L \ll L_d$ – scale of motion is much smaller than L_d – this means **we can drop the $\frac{1}{L_d^2}\psi'$ term in q**

(why? L_d is the minimum length-scale over which h can vary significantly in a geostrophically-balanced fluid; thus, $L < L_d$ means that h will not vary significantly over your length scale of interest L because gravity waves will smooth out variations in h):

$$q = \beta y + \zeta_g \tag{7}$$

(Note: this is identical to a pure *barotropic* (i.e. no vertical variations at all) QG system – i.e. one with a rigid lid at the top and thus no horizontal variations in fluid depth, and thus the governing equation reduces to conservation of absolute vorticity.)

where

$$\overline{q} = \beta y \tag{8}$$

$$q' = \zeta_g = \nabla^2 \psi' \tag{9}$$

governing equation: $\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0$
$v' = \frac{\partial \psi'}{\partial x}$
$\frac{\partial \overline{q}}{\partial y} = \beta$
Gives

$$\frac{\partial}{\partial t}(\nabla^2\psi') + U\frac{\partial}{\partial x}(\nabla^2\psi') + \beta\frac{\partial \psi'}{\partial x} = 0 \tag{10}$$

Seek wave solutions of form:

$$\psi' = Re\{\tilde{\psi}' e^{i(kx+ly-\omega t)}\} \tag{11}$$

Wave amplitude: $\tilde{\psi}'$
Wavenumber in x and y: k, l; total wavenumber $K^2 = k^2 + l^2$
Wave frequency: ω
Substitute in: $((-i\omega + iUk)((i^2)K^2) + \beta(ik))\tilde{\psi}' = 0$
$((-\omega + Uk)(-K^2) + \beta k)\tilde{\psi}' = 0$
Yields the **dispersion relation for barotropic** (i.e. uniform in height) **Rossby waves**:

$$\boxed{\omega = Uk - \frac{\beta k}{K^2}} \tag{12}$$

Rossby wave phase speed (i.e. velocity of individual wave):

$$c^x = \frac{\omega}{k} = U - \frac{\beta}{K^2} \tag{13}$$

$$c^y = \frac{\omega}{l} = U\frac{k}{l} - \frac{k}{l}\left(\frac{\beta}{K^2}\right) \tag{14}$$

- zero mean flow ($U = 0$): phase speed is **westward**; U simply doppler-shifts the waves
- dispersive: longer wavelength waves travel faster
- wave with wavelength K is stationary if $U = \frac{\beta}{K^2}$

Rossby wave group velocity (i.e. velocity of wave energy and information / packet of waves):

$$\boxed{c_g^x = \frac{\partial \omega}{\partial k} = U + \frac{\beta(k^2 - l^2)}{K^4}} \tag{15}$$

$$\boxed{c_g^y = \frac{\partial \omega}{\partial l} = \frac{2kl\beta}{K^4}} \tag{16}$$

Note: can write c_g^x in terms of c_x:

$$c_g^x = c^x + \frac{2\beta k^2}{K^4} \tag{17}$$

Thus, $c_g^x > c^x$ always: wave energy (packets) moves eastward faster than individual waves (e.g. if wave is stationary, wave energy still propagates eastward)

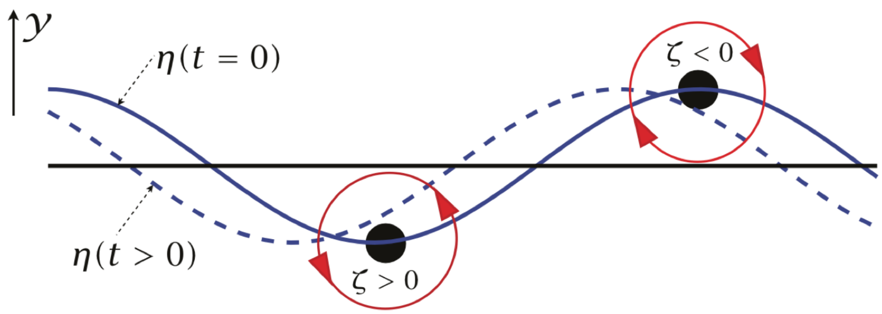

Fig. 6.6

Figure 1: VallisE Fig 6.3: mechanism for Rossby wave propagation, which can be understood from PV conservation.

3.1 Mechanistic understanding of westward propagation of Rossby Waves: VallisE FIG 6.3

GROUP: EXPLAIN THIS

- Imagine stationary zonal line of fluid parcels, $q = \beta y + \zeta_g$ (take infinite L_d for simplicity – i.e. ignore variations in h)
- disturbance displaces some parcels northward
- conservation of q: y increases $\rightarrow \zeta_g$ decreases – i.e. negative relative vorticity is generated
- induced clockwise flow displaces parcels to the west of initial disturbance northward – westward propagation!

This mechanism is the same for a continuously-stratified fluid, too. (See p.235 summary box)

Essential ingredient for Rossby waves: a background gradient in potential vorticity.

On Earth, this is dominated by the background gradient in planetary vorticity, β. Other examples: vortex Rossby waves on a hurricane (strong radial gradient in relative vorticity).

Remember: this mechanistic view only explains the *westward* propagation of Rossby waves... don't forget that Rossby waves also propagate meridionally as well as (for a stratified fluid) vertically.

4 Flow with finite L_d

For scales of motion close to or larger than L_d, we need to keep the full equation of q (i.e. allow for variations in h).

The deformation wavenumber is simply the inverse of the deformation radius: $k_d = \frac{1}{L_d}$

$$q = \beta y + \zeta_g - k_d^2 \psi' \tag{18}$$

Again, constant zonal flow basic state: $\overline{\psi} = -Uy$

QGPV basic state and perturbation:

$$\overline{q} = \beta y + k_d^2 (Uy) \tag{19}$$

$$q' = \zeta_g - k_d^2 \psi' = \nabla^2 \psi' - k_d^2 \psi' \tag{20}$$

governing equation: $\frac{\partial q'}{\partial t} + U \frac{\partial q'}{\partial x} + v' \frac{\partial \overline{q}}{\partial y} = 0$

Gives

$$(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x})(\nabla^2 \psi' - k_d^2 \psi') + (\beta + U k_d^2) \frac{\partial \psi'}{\partial x} = 0 \tag{21}$$

Wave solution: $\psi' = Re\{\tilde{\psi'} e^{i(kx+ly-\omega t)}\}$

Yields **dispersion relation**:

$$\boxed{\omega = Uk - k \frac{\beta + U k_d^2}{K^2 + k_d^2}} \tag{22}$$

Wave phase speed:

$$c^x = \frac{\omega}{k} = U - \frac{\beta + U k_d^2}{K^2 + k_d^2} \tag{23}$$

$$c^y = \frac{\omega}{l} = U \frac{k}{l} - \frac{k}{l} \left(\frac{\beta + U k_d^2}{K^2 + k_d^2} \right) \tag{24}$$

Now U no longer simply doppler shifts the zero-mean-flow (i.e. $U = 0$) wave speed – things are more complex.

Group velocity:

$$\boxed{c_g^x = \frac{\partial \omega}{\partial k} = U + \frac{(\beta + U k_d^2)(k^2 - l^2 - k_d^2)}{\left(K^2 + k_d^2\right)^2}} \tag{25}$$

$$\boxed{c_g^y = \frac{\partial \omega}{\partial l} = \frac{2kl(\beta + U k_d^2)}{(K^2 + k_d^2)^2}} \tag{26}$$

For $U = 0$, cases:

- Sufficiently long horizontal waves (i.e. small k; $k^2 < l^2 + k_d^2$): $c_g^x < 0$
- Shorter waves: $c_g^x > 0$

General properties of real Rossby waves:

- Long waves move energy/information westward, shorter waves move energy/information eastward
- Yet zonal phase speed is always westward relative to the mean flow.

5 Rossby waves in stratified QG flow: vertical propagation

See VallisE 6.4. With stratification, Rossby waves can have a *vertical* wavelength (wavenumber) too and they can propagate vertically in addition to horizontally. Mathematically this would show up by accounting for a vertical wavenumber, m, in the derivation.

6 Compare and contrast: gravity waves vs. Rossby waves

Gravity waves:

- **Oscillation**: vertical
- **Propagation**: principally horizontal (x,y); also vertical (z) in stratified flow
- **Required fluid property**: stratification (decrease in density with height)
- **Restoring mechanism**: gravity
- **Nature of dynamics**: unbalanced

Rossby waves:

- **Oscillation**: horizontal, perpendicular to PV contours (i.e. along PV gradient)
- **Propagation**: principally parallel to PV contours, with high PV to the right (think: on Earth, f increases moving poleward), but perpendicular to PV contours too; also vertical (z) in stratified flow
- **Required fluid property**: horizontal PV gradient
- **Restoring mechanism**: PV conservation
- **Nature of dynamics**: departure from balanced (geostrophic)

Note: the QG system actually has filtered out unbalanced motions (i.e. gravity waves, also sound waves). They still exist in reality!

To learn about how to consider both Rossby and gravity waves in a single system, see the next lecture notes. There we generalize the above derivation to also allow for unbalanced motions (gravity waves). We'll think about this first on the mid-latitude beta plane, and then on the equatorial beta plane.