EAPS 53600, Spring 2020
Lec 07: Mid-latitude beta plane dynamics (see Vallis big book 8.1-8.2)

1 Intro

The derivation for Rossby waves in the previous lecture focused on a flow that is in near-geostrophic balance (using quasi-geostrophic PV). This means that it does not allow for – i.e. it has **filtered out** – any unbalanced flow. Thus, you get Rossby waves, but *not* gravity waves.

Let's now generalize this derivation to allow for (unbalanced) gravity waves, too. We will do this by combining the governing equation for QG PV with the original governing equations for the height and flow fields as well.

This is the most general (and complicated) form of the equations on a "midlatitude" beta plane – i.e. where $|\beta y| \ll f_0$. (See next lecture for the equatorial beta plane, where $f_0 = 0$).

2 General derivation

In a system, Rossby and gravity waves may co-exist – requires: 1) PV gradient and 2) stratification .

Linearized single layer rotating SW system on beta plane ($f = f_0 + \beta y$):

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} \quad (EQ1) \tag{1}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y} \quad (EQ2) \tag{2}$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad (EQ3) \tag{3}$$

First, we can combine these:

$\frac{\partial}{\partial y}$ (1): $\frac{\partial}{\partial t}\frac{\partial u}{\partial y} - (f\frac{\partial v}{\partial y} + v\frac{\partial f}{\partial y}) = -g\frac{\partial^2 h}{\partial x \partial y}$

$\frac{\partial}{\partial x}$ (2): $\frac{\partial}{\partial t}\frac{\partial v}{\partial x} + f\frac{\partial u}{\partial x} = -g\frac{\partial^2 h}{\partial y \partial x}$

Subtract (bottom - top), $\frac{\partial f}{\partial y} = \beta$: $\frac{\partial \zeta_g}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$

Use (3) to eliminate divergence: $\frac{\partial \zeta_g}{\partial t} - \frac{f}{H}\frac{\partial h}{\partial t} + \beta v = 0$

Combine derivatives:

$$\frac{\partial}{\partial t}\left(\zeta_g - \frac{f}{H}h\right) + \beta v = 0 \quad (EQ4) \tag{4}$$

This is **almost the same linearized QGPV equation we derived in the previous section** (for zero mean flow $\overline{u} = 0$, and $\frac{\partial \overline{q}}{\partial y} = \beta$). **The one key difference: we are keeping f general (rather than setting it to $f = f_0$) in the term multiplying h.**

Next

$\frac{f}{gH}\frac{\partial}{\partial t}$ (1): $\frac{f}{gH}u_{tt} - \frac{f^2}{gH}v_t = -\frac{f}{H}h_{xt}$
$\frac{1}{gH}\frac{\partial^2}{\partial t^2}$ (2): $\frac{1}{gH}v_{ttt} + \frac{f}{gH}u_{tt} = -\frac{1}{H}h_{ytt}$
$\frac{1}{H}\frac{\partial^2}{\partial t \partial y}$ (3): $\frac{1}{H}h_{tty} + (u_{txy} + v_{tyy}) = 0$
$\frac{\partial}{\partial x}$ (4): $v_{txx} - u_{txy} - \frac{f}{H}h_{tx} + \beta v_x = 0$
Combine above equations: (6 - (5+7+8))
$(\frac{1}{gH}v_{ttt}+\frac{f}{gH}u_{tt}) - \Big((\frac{f}{gH}u_{tt} - \frac{f^2}{gH}v_t) + (\frac{1}{H}h_{tty} + u_{txy} + v_{tyy}) + (v_{txx} - u_{txy} - \frac{f}{H}h_{tx} + \beta v_x)\Big) = (-\frac{1}{H}h_{ytt}) - \big((-\frac{f}{H}h_{tx}) + 0 + 0\big)$
Terms that cancel: h, $\frac{f}{gH}u_{tt}$, u_{txy}. This yields

$$\boxed{\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{f^2}{c^2}\frac{\partial v}{\partial t} - \frac{\partial}{\partial t}\left(\nabla^2 v\right) - \beta\frac{\partial v}{\partial x} = 0} \tag{5}$$

where $c = c_{GW} = \sqrt{gH}$

This equation is at the heart of the complete set of wave dynamics in a medium with varying f (e.g. a beta plane).

This complicated equation has non-constant coefficients – f varies with y. Hence, in its current form you cannot simply assume pure wave solutions...

2.1 Option 1: constant $f = f_0$ – the mid-latitude beta-plane

First, let's start with a simplified version. **Let's try taking $f = f_0$.** HOWEVER, **we still retain** β (to allow for Rossby waves)! So we are taking f constant except where differentiated. This may seem weird! But basically we want to retain the dynamics that depend on gradients in f (Rossby waves) but then neglect the small variations in f for the dynamics that care only about f itself (gravity waves).

It is also a very convenient mathematical simplification that allows us to solve for regular waves in this system. With constant coefficients, we can assume plane-wave solutions in both the x and y directions of the form

$$v = \tilde{v}e^{(i(kx+ly-\omega t))} \tag{6}$$

Plugging this in yields
$\frac{1}{c^2}(-i\omega)^3 + \frac{f_0^2}{c^2}(-i\omega) - (-i\omega)((ik)^2 + (il)^2) - \beta(ik) = 0$
$\frac{1}{c^2}\omega^3 - \frac{f_0^2}{c^2}\omega - \omega(k^2 + l^2) - \beta k = 0$
which can be written as a cubic equation in ω (cubic as expected from equation with third-order time derivative).

Thus, the **dispersion relation** for this wave system is

$$\boxed{\omega^2 - \frac{c^2\beta k}{\omega} = f_0^2 + c^2(k^2 + l^2)} \tag{7}$$

Cases:
1) f-plane ($\beta = 0$)

$$\omega(\omega^2 - f_0^2 - c^2(k^2 + l^2)) = 0 \tag{8}$$

We saw this before:
$\omega = 0$: geostrophic flow (time-independent)
$\omega^2 = f_0^2 + c^2K^2$: Poincare waves
No Rossby waves. (Why? $\beta = 0$)
2) High frequency waves $\omega \gg f_0$ (requires $\omega \gg \beta$ too for physicality)

$$\omega^2 - c^2(k^2 + l^2) = 0 \tag{9}$$

$\omega = cK$: regular non-rotating gravity waves
3) Low frequency waves $\omega \ll f_0$
i.e. $\omega^2 \ll f_0^2$
$\omega = \frac{-\beta k c^2}{c^2(k^2+l^2)+f_0^2}$

$$\omega = \frac{-\beta k}{k^2 + l^2 + k_d^2} \tag{10}$$

where $k_d^2 = \frac{f_0^2}{c^2}$
QG Rossby waves
This system has it all!

Notice: there is a distinction here between waves according to frequency: gravity waves are high frequency, Rossby waves are low frequency.

2.2 Visualizing the dispersion relation

Let's visualize this dispersion relation and its associated wave solutions.

$$\omega^2 - \frac{c^2\beta k}{\omega} = f_0^2 + c^2(k^2 + l^2) \tag{11}$$

Nondimensionalization:

- time: $T \sim f_0^{-1}$
- distance: $L \sim L_d = k_d^{-1} = \frac{c}{f_0}$
- velocity: $U \sim \frac{L}{T} = c$ (this is consistent with intuition!)

Recall: $c = \sqrt{gH}$, $L_d = \frac{c}{f_0}$, $k_d = \frac{f_0}{c}$
Thus:

- frequency: $\omega = \hat{\omega} f_0$ (units: $[\frac{1}{T}]$)
- wavenumber: $(k, l) = (\hat{k}, \hat{l})k_d$ (units: $[\frac{1}{L}]$)
- β: $\beta = \hat{\beta} f_0 k_d$ (units: $[\frac{1}{TL}]$) – from $\frac{df}{dy}$

Plug these in
$(\hat{\omega} f_0)^2 - \frac{c^2 \hat{\beta} f_0 k_d \hat{k} k_d}{\hat{\omega} f_0} = f_0^2 + c^2((\hat{k} k_d)^2 + (\hat{l} k_d)^2)$
Plug in for $k_d = \frac{f_0}{c}$: $(\hat{\omega} f_0)^2 - \frac{c^2 \hat{\beta} f_0 \hat{k} (\frac{f_0}{c})^2}{\hat{\omega} f_0} = f_0^2 + c^2 (\frac{f_0}{c})^2 ((\hat{k})^2 + (\hat{l})^2)$
Divide through by f_0^2, and all c and f_0 terms cancel, yielding

$$\hat{\omega}^2 - \hat{\beta}\frac{\hat{k}}{\hat{\omega}} = 1 + (\hat{k}^2 + \hat{l}^2) \tag{12}$$

Now we have an equation relating frequency and wavenumber that contains only a single external parameter: $\hat{\beta}$
$\hat{\beta} = \frac{\beta L_d}{f_0}$
Characteristic values for the the atmosphere: 40 ^oN
$L_d = 1000\ km = 10^6\ m$
$f_0 = 2\Omega sin(\phi) \approx 10^{-4}\ s^{-1}$
$\beta = \frac{2\Omega cos(\phi)}{a_{Earth}} \approx 2 * 10^{-11}\ m^{-1} s^{-1}$
Thus $\hat{\beta} = \frac{2*10^{-11}\ m^{-1}s^{-1}}{10^{-4}\ s^{-1}}(10^6\ m) = (2 * 10^-7\ m^{-1})(10^6\ m) = 0.2$
Let's take $\hat{\beta} = 0.2$:
DRAW ME!
(Reminder: Standard convection is to define frequency (ω) to be non-negative, i.e. $\omega \geq 0$. Wavenumber (k, l, m) may be negative or positive; positive = eastward/northward/upward; negative = westward/southward/downward. Why? The wavevector is $\mathbf{k} \cdot \mathbf{x}$, where $\mathbf{x}$ is defined positive eastward/northward/upward. Thus $\mathbf{k} \cdot \mathbf{x} < 0$ means that e.g. for zonal motion $k < 0$ points opposite to the positive x direction (eastward) and hence the wave will move westward.)

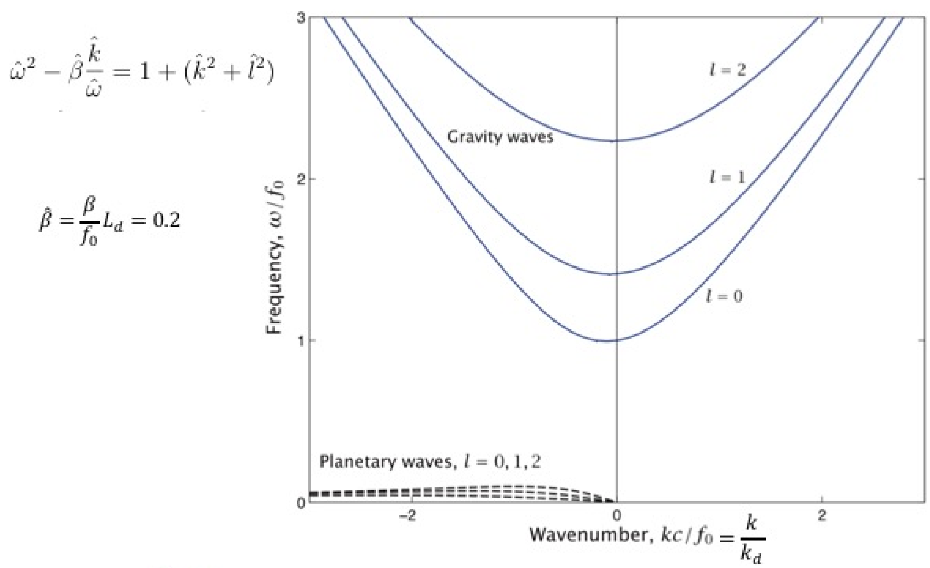

Fig. 8.2

You can see the strong separation in waves between low and high frequency. Let's understand these:

1) Higher frequency waves $\hat{\omega} \gg \hat{\beta}$: Poincare waves

$$\hat{\omega}^2 = 1 + (\hat{k}^2 + \hat{l}^2) \tag{13}$$

So, for example:

- $l = 0$ (i.e. constant in the y-direction), this yields: $\hat{\omega}^2 = 1 + \hat{k}^2$ – i.e. a curve that is linear at larger $\hat{k}$ with a minimum at $(\hat{k}, \hat{\omega}) = (0, 1)$, which corresponds to $\omega = f_0$ as the lowest possible frequency of a Poincare wave – this is the MINIMUM possible GW frequency

- $l = 1$: meridional wavelength = deformation length scale, the wave has a higher frequency and wave speed

3) Low frequency waves $\hat{\omega} \ll 1$: Rossby waves
$\omega = \frac{-\hat{\beta}\hat{k}}{(\hat{k}^2+\hat{l}^2)+f_0^2}$

$$\hat{\omega} = \frac{-\hat{\beta}\hat{k}}{\hat{k}^2 + \hat{l}^2 + 1} \tag{14}$$

So, for example:

- $l = 0$, this yields: $\hat{\omega} = \frac{-\hat{\beta}\hat{k}}{\hat{k}^2+1}$. – this is the MAXIMUM possible RW frequency (Note $\hat{k} < 0$ required for this to be real)