EAPS 53600, Spring 2020
Lec 08: Equatorial beta plane dynamics (Vallis Big Book 8.2)

1 Intro

The previous lecture found shallow-water waves on the mid-latitude beta plane, where $f = f_0$ but we still retain β.

Let's try a second option: the "equatorial" beta plane, where $f_0 = 0$ and $f = \beta y$.

1.1 Option 2: $f_0 = 0$, $f = \beta y$ – the equatorial beta-plane

We'll start again with the governing equation we derived last time for the linearized single layer rotating SW system on beta plane:

$$\boxed{\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{f^2}{c^2}\frac{\partial v}{\partial t} - \frac{\partial}{\partial t}\left(\nabla^2 v\right) - \beta\frac{\partial v}{\partial x} = 0} \tag{1}$$

Equatorial beta plane: $f_0 = 0$, i.e. $f = \beta y$

This system now has *non-constant* coefficients - since βy depends on y!

Thus, we search for solutions with a plane wave in the *zonal direction only*, i.e.

$$v = \tilde{v}(y)e^{(i(kx-\omega t))} \tag{2}$$

with boundary conditions: $\tilde{v}(y) \to 0$ as $y \to \pm\infty$.

Substitution yields:

$$\frac{d^2\tilde{v}}{dy^2} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right)\tilde{v} = 0 \tag{3}$$

(Note: $\frac{d^2\tilde{v}}{dy^2}$ comes from the Laplacian)

We again want to non-dimensionalize this solution. For our equatorial beta plane $f_0 = 0$, which means we need a different time scale.

1.1.1 Deformation radius on the equatorial beta-plane

The f-plane deformation radius: $L_d = \frac{c}{f}$ = distance a gravity wave travels before feeling the effect of rotation.

Same concept applies at low latitudes.

$f = \beta y$ and take $y = L_d$ we get

$L_d = \frac{c}{\beta L_d}$

Equatorial beta-plane deformation radius:

$$L_{d,eq} = \sqrt{\frac{c}{\beta}} \tag{4}$$

Initially no rotation, but gradually feels increasingly strong rotation moving poleward. This also yields an intrinsic timescale for the system from $T = UL = cL_{d,eq}$

$$T \sim \left(\sqrt{c\beta}\right)^{-1} \tag{5}$$

The velocity scale c is the same.

1.2 Derivation (see Vallis Big Book 8.2.1)

Feel free to skip the solution in the next section

Non-dim:

- $\hat{\omega} = \frac{\omega}{\sqrt{c\beta}}$
- $\hat{k} = \frac{k}{\sqrt{\frac{c}{\beta}}}$
- $\hat{y} = \frac{y}{\sqrt{\frac{\beta}{c}}}$

$$\frac{d^2\tilde{v}}{d\hat{y}^2} + \left(\hat{\omega}^2 - \hat{k}^2 - \frac{\hat{k}}{\hat{\omega}} - \hat{y}^2\right)\tilde{v} = 0 \tag{6}$$

Put into a standard form using

$$\tilde{v}(\hat{y}) = \Psi(\hat{y})e^{-\frac{\hat{y}^2}{2}} \tag{7}$$

yields Hermite's equation:

$$\frac{d^2\Psi}{d\hat{y}^2} - 2\hat{y}\frac{d\Psi}{d\hat{y}} + \lambda\Psi = 0 \tag{8}$$

where

$$\lambda = \hat{\omega}^2 - \hat{k}^2 - \frac{\hat{k}}{\hat{\omega}} - 1 \tag{9}$$

Hermite's equation is an eigenvalue equation, such only quantized solutions exist where

$$\lambda = 2m \qquad m = 0, 1, 2, ... \tag{10}$$

and these solutions are Hermite polynomials

$$\Psi(\hat{y}) = H_m(\hat{y}) \tag{11}$$

where $H_0 = 1$, $H_1 = 2\hat{y}$, $H_2 = 4\hat{y}^2 - 2$, $H_3 = 8\hat{y}^3 - 12\hat{y}$,...
(**FIG8-5**)
Note: $H_m(-\hat{y}) = (-1)^m H_m(\hat{y})$
Thus, the set of solutions for $\tilde{v}$ is:

$$\tilde{v}_m(\hat{y}) = H_m(\hat{y})e^{-\frac{\hat{y}^2}{2}} \qquad m = 0, 1, 2, ... \tag{12}$$

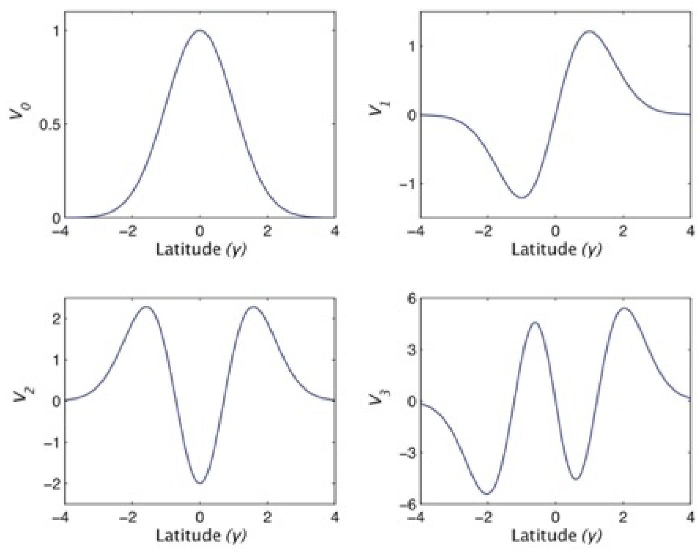

Fig. 8.5

where $H_0 = 1$, $H_1 = 2\hat{y}$, $H_2 = 4\hat{y}^2 - 2$, $H_3 = 8\hat{y}^3 - 12\hat{y}$,...

Note: H is identical for $\pm y$ when m is even. Since H is used for solutions for v, v is symmetric about the equator (e.g. northward for $y > 0$ and southward for $y < 0$) for m *odd*. Meanwhile, u and η are symmetric about the equator for m even.

The **dispersion relation** comes directly from $\lambda = 2m$, i.e.

$2m = \hat{\omega}^2 - \hat{k}^2 - \frac{\hat{k}}{\hat{\omega}} - 1$

which rearranges to

$$\hat{\omega}^2 - \frac{\hat{k}}{\hat{\omega}} = 2m + 1 + \hat{k}^2 \qquad m = 0, 1, 2, ... \tag{13}$$

1.3 Solution

Recall, the dispersion relation for our case where we took $f = f_0$

$f = f_0$: $(\hat{\omega} = \frac{\omega}{f_0}$; $(\hat{k}, \hat{l}) = \frac{(k,l)}{k_d}$; $\hat{\beta} = \frac{\beta}{f_0 k_d})$

$$\omega^2 - \frac{c^2 \beta k}{\omega} = f_0^2 + c^2(k^2 + l^2) \tag{14}$$

and in non-dimensional form

$$\hat{\omega}^2 - \hat{\beta}\frac{\hat{k}}{\hat{\omega}} = 1 + (\hat{k}^2 + \hat{l}^2) \tag{15}$$

For the equatorial beta-plane ($f = \beta y$), the dispersion relation is: (**see formal derivation below**)

$$\omega^2 - \frac{c^2 \beta k}{\omega} = (2m + 1)c\beta + c^2 k^2 \qquad m = 0, 1, 2, ... \tag{16}$$

and in non-dimensional form $f = \beta y$: ($\hat{\omega} = \frac{\omega}{\sqrt{c\beta}}$; $\hat{k} = \frac{k}{\sqrt{\frac{c}{\beta}}}$)

$$\hat{\omega}^2 - \frac{\hat{k}}{\hat{\omega}} = 2m + 1 + \hat{k}^2 \qquad m = 0, 1, 2, ... \tag{17}$$

$m = 0, 1, 2, ...$ appears because the solution to the governing equation is a **set of quantized harmonic solutions**. m is very closely analogous to l, i.e. higher m corresponds to higher-order variability in the y-direction!

- m even = v symmetric about the equator
- m odd = v anti-symmetric about the equator

Thus, the only changes are:

- l^2 disappears (because no longer plane wave in y-direction)
- instead m appears – a *set* of harmonic solutions in the y-direction, very closely analogous to l
- first term RHS ($f_0^2 \to (2m+1)c\beta$; non-dim: $1 \to 2m+1$)
- $\hat{\beta}$ does not appear – zero external parameters!

Visualization: (**FIG8-6**)

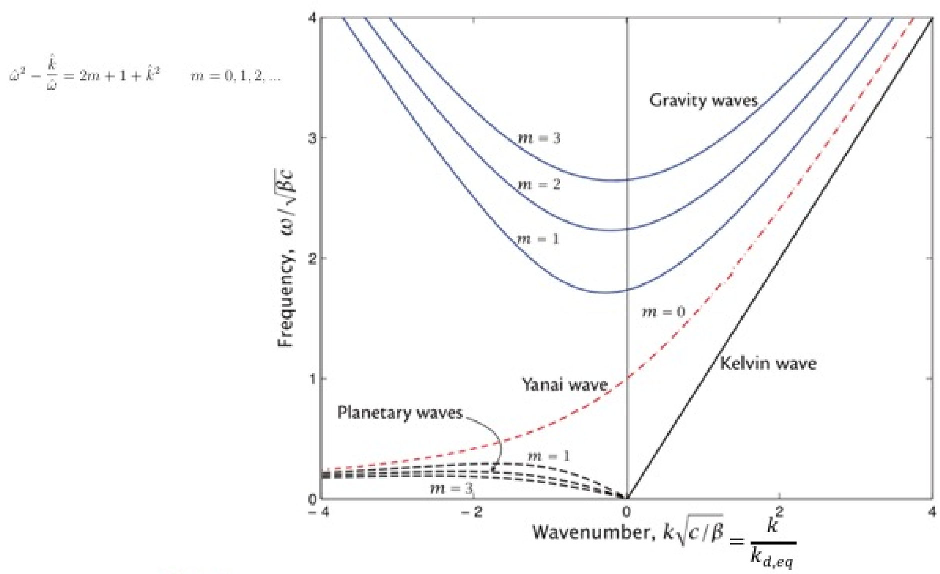

Fig. 8.6

Note: "planetary waves" = Rossby waves.

1.3.1 High frequency waves

Neglect $\frac{c^2\beta k}{\omega}$ (nondim: $\frac{\hat{k}}{\hat{\omega}}$)

Yields:

$\omega^2 = (2m+1)c\beta + c^2k^2$ ($\hat{\omega}^2 = 2m+1+\hat{k}^2$)

These are equatorially-trapped Poincare waves (f replaced with $c\beta$) – zero meridional propagation

For sufficiently high wavenumbers, this is just regular non-rotating SW gravity waves.

1.3.2 Low frequency waves

Neglect ω^2 (nondim: $\hat{\omega}^2$)

Yields:

$\omega = \frac{-\beta k}{(2m+1)\frac{\beta}{c}+k^2}$ ($\hat{\omega} = \frac{-\hat{k}}{2m+1+\hat{k}^2}$)

These are equatorially-trapped zonally-propagating Rossby waves – zero meridional propagation.

Note: again a large frequency gap between low-frequency Rossby waves and high frequency gravity waves. Can show theoretically (V p.309) that the ratio of the minimum GW frequency and maximum RW frequency is given by

$$\frac{\omega_{GW,min}}{\omega_{RW,max}} = 2(2m+1) \tag{18}$$

This is independent of both β and c! – it is a universal property of a fluid on any equatorial beta plane.

1.3.3 Special case 1: m=0 – Yanai wave (mixed Rossby-Gravity wave)

$\hat{\omega}^2 - \frac{\hat{k}}{\hat{\omega}} = 1 + \hat{k}^2$

This has two solutions:

$$\omega = -ck \tag{19}$$

$$\omega = \frac{kc}{2} \pm \frac{1}{2}\sqrt{k^2c^2 + 4c\beta} \tag{20}$$

The first solution is non-physical: it is a westward-propagating gravity wave, which will move away from the equator and blow up. This is fundamentally different from the $f = f_0$ case earlier, as this blow-up only occurs because we require solutions to decay moving away from $y = 0$.

Second solution cases:

- $k = 0$: $\omega = \sqrt{c\beta}$ ($(\hat{k}, \hat{\omega}) = (0, 1)$
- $k \to +\infty$: $\omega = ck$ – eastward-propagating gravity wave (high frequency!)
- $k \to -\infty$: $\omega = -\frac{\beta}{k}$ – westward-propagating Rossby wave (low frequency!)

1.3.4 Special case 2: m=-1 – Kelvin wave

Oddly, setting $m = -1$ in the dispersion relation yields another solution:

$\omega^2 - \frac{c^2\beta k}{\omega} = (2(-1)+1)c\beta + c^2k^2$

$\omega^2 - \frac{c^2\beta k}{\omega} = -c\beta + c^2k^2$

Which gives the solution

$$\omega = \pm ck \tag{21}$$

These are equatorially-trapped Kelvin waves, for which $\omega = +ck$ (i.e. eastward propagating) is the only physical solution.

1.4 Rossby adjustment

Analogous to geostrophic adjustment problem, but now on an equatorial beta plane.

SHOW MOVIES / SIMULATIONS

1.5 Summary

Equatorial beta-plane yields multiple types of waves:

- high frequency: Poincare (gravity) waves, multiple modes
- low frequency: Rossby waves (westward propagating), multiple modes
- mixed frequencies: Mixed Rossby-Gravity wave (Yanai wave), single mode – eastward propagating Poincare wave at high frequency, westward propagating Rossby wave at low frequency
- special case: Kelvin wave (single mode) – eastward propagating, "equatorially-trapped" (i.e. always being turned into the Equator by the Coriolis force, regardless of which hemisphere!)

2 Matsuno-Gill Model (Vallis Big Book 8.5)

Forced, steady problem: suppose a bump is permanently being recreated. What does the steady state solution look?

This problem has analytic solutions, called the "Matsuno-Gill Model". In this case, the forcing for a "bump" is thought of as a source of heating in the atmosphere. **See figures below.**

(**FIG8-11**)
(**FIG8-7**)
(**FIG18-23**)

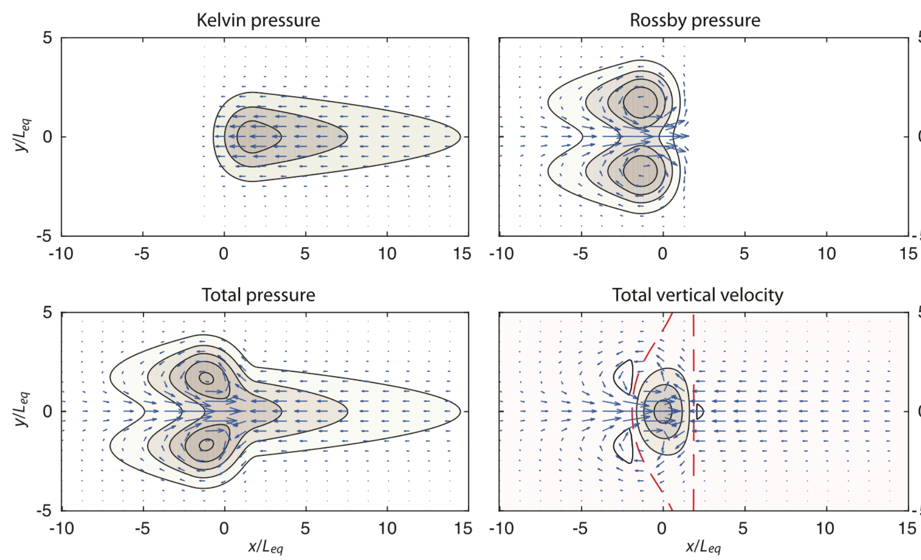

Fig. 8.11

"Gill model" (or "Matsuno-Gill" model) – see Vallis 8.5
Shading = pressure (shading = low), vertical velocity (shading = positive)

Figure 1: Matsuno-Gill model response to a steady source of heating localized at the equator (here at $(x, y) = (0, 0)$). Subplots show pressure and horizontal flow perturbations associated with Kelvin waves (top left) and Rossby waves (top right) and the sum of the two (bottom left); with total vertical velocity (bottom right; greys = upward motion) and total horizontal flow perturbations associated with the heating forcing.

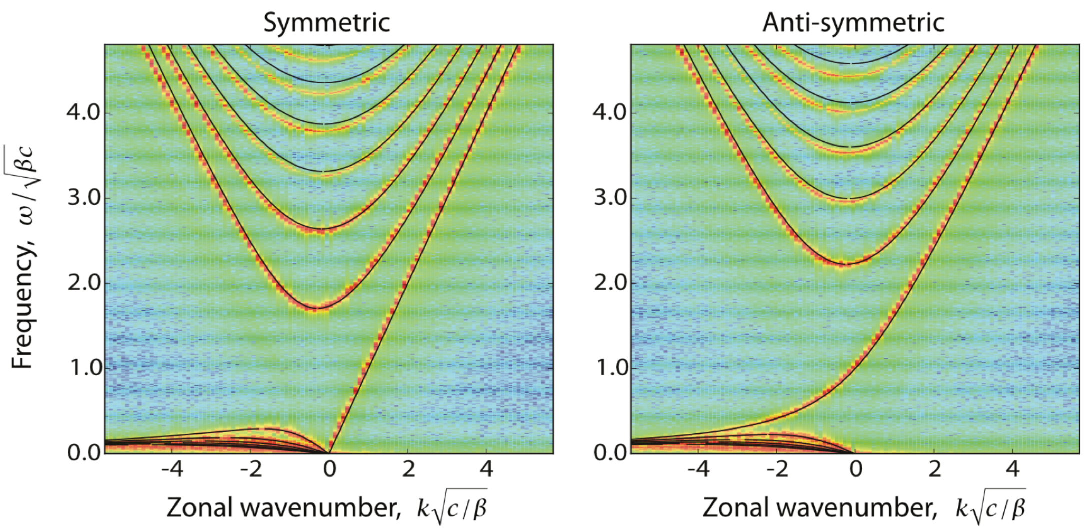

Fig. 8.7

Figure 2: Colors: Frequency-wavenumber plot for a one-layer shallow water model on the equatorial beta plane. Colors: power spectrum of fluid height (i.e. wave energy) as a function of ω and k . Black lines: theoretical predictions from the Matsuno-Gill model for the dispersion relations of equatorial Rossby and rotating gravity waves overlaid. Symmetric = same sign perturbation in NH and SH; Anti-symmetric = opposite signs.

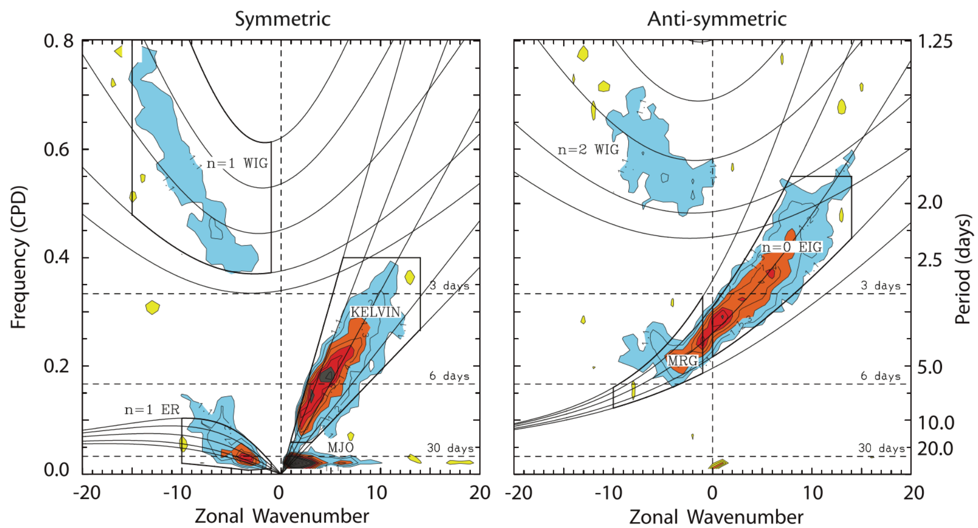

Fig. 18.23

Figure 3: Similar to previous plot, but for power spectrum of cloud brightness, estimated from satellite data in the tropics 15S-15N. Black lines show theoretical solutions for dispersion relations of equatorial Rossby waves, rotating gravity waves, and mixed rossby-gravity (Yanai) waves for different equivalent depths, h_{eq}, from 8-90m; best fit is $h_{eq} = 25\ m$. "MJO" is the Madden-Julian Oscillation. (cf. Wheeler and Kiladis (1999).