EAPS 53600, Spring 2020
Lec 09: Barotropic instability (VallisE 8.1-8.3)

March 3, 2020

1 Intro

In life, we experience the weather. **The weather is interesting because it is always changing – the atmosphere is *not* steady**. Instead, it is *unsteady* – varying in time.

Hydrodynamic states that could occur in nature:

- Steady (time-independent): zero motion (e.g. geostrophic balance)
- Time-dependent, stable: waves – perturbations that propagate (and eventually decay due to dissipation, e.g. friction at boundaries)
- Time-dependent, unstable: **instability – infinitesimally small perturbations *grow* with time**

Where we've gone and where we're going next:

- Up til now: balanced flow, waves (gravity, Rossby) in a single layer of fluid
- **Now: hydrodynamic instability of a flow** – many forms. We focus on two:

1. **Barotropic instability** – does not require density variations
   - medium: any fluid
   - essential property: **shear in flow** – spatial variations in wind speed/direction
   - relevance: jets, vortices; 2D and 3D turbulence; water coming out of your faucet (if it were purely laminar flow it would be transparent (e.g. a water fountain); if its opaque, it is *turbulent*)
2. **Baroclinic instability** – requires density variations
   - medium: rotating, stably-stratified fluid
   - essential property: **horizontal density (temperature) gradients** (which is accompanied by vertical shear via thermal wind balance!)
   - relevance: generates extratropical ("baroclinic") cyclones – i.e. **weather that directly affects our lives on a daily basis!**

It turns out that these are actually dynamically very similar.

What do they both require? Rossby waves moving in *opposite* directions within the flow ("counter-propagating Rossby waves"). For the right conditions, these waves can *amplify each other* – this is the physical mechanism for the instability.

Question: What do Rossby waves need to exist?

A PV gradient. We must need the PV gradient to *change sign* then.

GROUP ACTIVITY: you already have the tools to understand barotropic instability... consider *two* contours, each with a PV gradient, and close enough to each other that they can feel the other's perturbation circulations. Hypothesis: for instability to occur, PV gradient of the two contours must be *opposite signed*. (Hint: try out two contours each with the same signed PV gradient, then create a wave on each – what circulations does this induce? how would the circulations from one contour affect the other contour, and vice versa?)

Importantly: on a rotating sphere like Earth, $\beta > 0$ everywhere. So to have the PV gradient change sign, **we'll need to consider a basic state with gradients in *relative* vorticity – i.e. wind shear.**

2 Recall: Instability, mathematically

General wave-like solution (ψ can be any variable):

$$\boxed{\psi' = Re\{\tilde{\psi} e^{i(kx-\omega t)}\}} \tag{1}$$

Wave phase speed: $c = \omega/k = c_r + ic_i$ – real *and* imaginary parts
Which can be written in a way that a human can understand as:

$$\boxed{\psi' = \left(\tilde{\psi} e^{kc_i t}\right) cos(k(x - c_r t))} \tag{2}$$

- Exponential growth rate of wave amplitude: $\boxed{\sigma = kc_i}$ (positive = growing, negative = decaying; units: $[s^{-1}]$)
- Amplitude of wave: $\tilde{\psi}$

Thus the **phase speed, c, tells you everything about how a wave will evolve**:

1. **real part** (c_r) = wave **propagation** ($\omega = kc_r$)
2. **imaginary part** (c_i) = exponential **growth/decay** of wave amplitude ($\sigma = kc_i$).

3 Barotropic instability: parallel shear flow

Basics: **laminar (i.e. "smooth") flow with variations in wind speed/direction, a.k.a. wind shear, can sometimes break down into vortices**.
Examples: a jet, two layers of fluid moving in different direction – **Fig 8.2**.
What determines when this happens?
We'll work with the simplest model: 2D incompressible flow:

1. **shallow water system**
2. **constant fluid depth** – this also means $L_d \to \infty$ ($k_d = 0$)

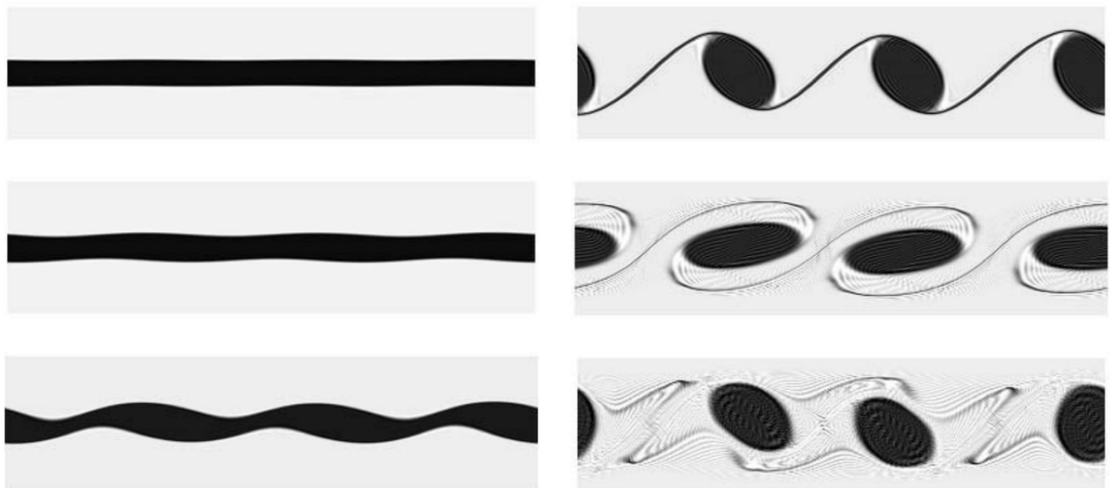

Fig. 8.2:

Figure 1: (Fig8-2) Breakdown of a vorticity band into coherent vortices in a nonlinear barotropic vorticity model. Shading = relative vorticity. Domain size is 4x1, with vorticity band width of 0.2 (arbitrary units). Initial condition is shown in FIG8-3 below.

3.1 Parallel shear flow (VallisE 8.2)

Recall PV conservation:

$$\frac{DQ}{Dt} = 0 \tag{3}$$

$$Q = \frac{\zeta + f}{h} \tag{4}$$

and its linearized form:

$$\frac{\partial Q'}{\partial t} + U\frac{\partial Q'}{\partial x} + v'\frac{\partial \overline{Q}}{\partial y} = 0 \tag{5}$$

Constant fluid depth: PV $\rightarrow$ absolute vorticity conservation ($Q \rightarrow (\zeta + f)$)

$$\frac{\partial \zeta'}{\partial t} + U\frac{\partial \zeta'}{\partial x} + v'\left(\frac{\partial \overline{\zeta}}{\partial y} + \beta\right) = 0 \tag{6}$$

Basic state: "**parallel shear flow**" – parallel flow, purely x-direction but **varying in y-direction**, i.e.

$$\overline{\mathbf{u}} = U(y)\hat{\mathbf{i}} \tag{7}$$

Thus, $\overline{\zeta} = \frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y}$, i.e

$$\boxed{\overline{\zeta} = -\frac{\partial U}{\partial y}} \tag{8}$$

This yields

$$\boxed{\frac{\partial \zeta'}{\partial t} + U\frac{\partial \zeta'}{\partial x} + v'\left(\beta - \frac{\partial^2 U}{\partial y^2}\right) = 0} \tag{9}$$

Now we are considering ***two*** **components to meridional PV gradient**:

- β : **planetary vorticity gradient**
- $-\frac{\partial^2 U}{\partial y^2}$: ***relative*** **vorticity gradient** ($\frac{\partial \overline{\zeta}}{\partial y}$)

Incompressible flow: can define u and v in terms of streamfunction

$$u' = -\frac{\partial \psi'}{\partial y} \tag{10}$$

$$v' = \frac{\partial \psi'}{\partial x} \tag{11}$$

$$\zeta' = \nabla^2 \psi' \tag{12}$$

(satisfies mass continuity $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$)
Plugging in yields

$$\frac{\partial \nabla^2 \psi'}{\partial t} + U\frac{\partial \nabla^2 \psi'}{\partial x} + \left(\beta - \frac{\partial^2 U}{\partial y^2}\right)\frac{\partial \psi'}{\partial x} = 0 \tag{13}$$

Coefficients depend on y, but not x: harmonic (wave) solutions in x, retain arbitrary y dependence, i.e. solution

$$\psi' = Re\{\tilde{\psi}(y)e^{ik(x-ct)}\} \tag{14}$$

and similarly

$$u' = \tilde{u}(y)e^{ik(x-ct)} \tag{15}$$

$$v' = \tilde{v}(y)e^{ik(x-ct)} \tag{16}$$

$$\zeta' = \tilde{\zeta}(y)e^{ik(x-ct)} \tag{17}$$

Plugging in for ψ' gives (subscripts = derivatives) – (note: this is easier to see the math)

$$u' = -\tilde{\psi}_y e^{ik(x-ct)} \tag{18}$$

$$v' = ik\tilde{\psi}e^{ik(x-ct)} \tag{19}$$

$$\zeta' = \left(-k^2\tilde{\psi} + \tilde{\psi}_{yy}\right)e^{ik(x-ct)} \tag{20}$$

Plugging these in to original equation gives
$(-ikc)((ik)^2\tilde{\psi} + \tilde{\psi}_{yy}) + U(ik)((ik)^2\tilde{\psi} + \tilde{\psi}_{yy}) + (ik)\tilde{\psi}\,(\beta - U_{yy}) = 0$
Divide by ik and combine first two terms:
$(U - c)((ik)^2\tilde{\psi} + \tilde{\psi}_{yy}) + \tilde{\psi}\,(\beta - U_{yy}) = 0$
$i^2 = -1$
Yields **Rayleigh-Kuo equation**:

$$\boxed{(U - c)(\tilde{\psi}_{yy} - k^2\tilde{\psi}) + (\beta - U_{yy})\tilde{\psi} = 0} \tag{21}$$

This equation governs the stability of the flow – do waves grow or decay? Depends on nature of c

$$c = c_r + ic_i \tag{22}$$

Possible outcomes:

- c purely real ($c_i = 0$): c is simply phase speed of a wave
- $c_i < 0$: wave decay exponentially
- $c_i > 0$: wave grows exponentially – **unstable**

Analytic solution for arbitrary U_{yy} is very difficult. We will focus on the simplest case below – a "piecewise-linear" flow field.

4 Rossby edge wave: zonal point jet

4.1 Flow with a sharp edge

Piecewise-linear zonal flow: U_y constant in intervals but with abrupt change at one or more y-discontinuities.

Simplest case: a **zonal point-jet** – one discontinuity (**FIG 8.4**)

GROUP: Do you think there are PV gradients here? If yes, where and what is its sign?

GROUP: Which direction will Rossby waves propagate at each interface?

Curvature $U_{yy} = 0$ everywhere – *except at y-discontinuities.*
Thus, $\frac{\partial \overline{Q}}{\partial y} = 0$ everywhere except at these interfaces.
Thus, since $\frac{\partial \overline{Q}}{\partial y}$ is required for Rossby waves, RWs may exist only at these interfaces!

Basic state: two regions

- $y > 0$: $U_1 = U_0 - Ay$ (above)
- $y < 0$: $U_2 = U_0 + Ay$ (below)

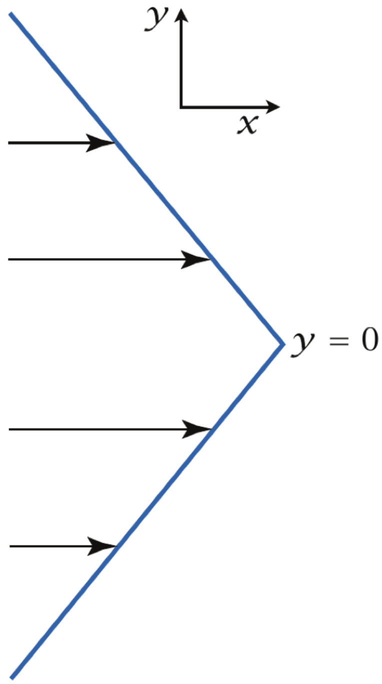

Figure 2: (Fig 8-4). A point jet: piecewise-linear flow with constant positive shear to the south and constant negative shear to the short – one discontinuity.

where A is a positive constant.
Thus, at $y = 0$: $U_1(y = 0) = U_2(y = 0) = U_0$ (constant)
At interface ($y = 0$):

- velocity (U): continuous
- vorticity ($-U_y$): discontinuous

Either side of interface (i.e. $y \neq 0$): $U_{yy} = 0$, thus

$$(U - c)(\tilde{\psi}_{yy} - k^2\tilde{\psi}) = 0 \tag{23}$$

For $c \neq U$, the solution is:

$$\tilde{\psi} = \begin{cases} \Phi_1 e^{-ky} & \text{if } y > 0 \\ \Phi_2 e^{ky} & \text{if } y < 0 \end{cases}$$

These decay moving away from interface on both sides.

4.2 Matching solutions across any sharp edge

What about at the vorticity discontinuity $y = 0$ itself? How do we match our solutions across this discontinuity?

Solve linearized equation separately in each interval, matching adjacent solutions at the discontinuity points.

Two matching conditions:

1. pressure continuous across interface
2. normal velocity at interface consistent with motion of interface

1) Pressure continuity
Linearized momentum equation in the direction along the interface (x):

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \tag{24}$$

Wave-like solutions in x-direction:

$$u' = \tilde{u}(y)e^{ik(x-ct)} \tag{25}$$
$$v' = \tilde{v}(y)e^{ik(x-ct)} \tag{26}$$
$$p' = \tilde{p}(y)e^{ik(x-ct)} \tag{27}$$

Plugging in yields:

$$ik(U-c)\tilde{\psi}_y - ik\tilde{\psi}U_y = -\frac{ik}{\rho_0}\tilde{p} \tag{28}$$

Since we require $\tilde{p}$ continuous across the interface, this implies:

$$\boxed{\Delta\left[(U-c)\tilde{\psi}_y - \tilde{\psi}U_y\right] = 0} \tag{29}$$

2) Material interface continuity
Normal velocity at (zonal) interface:

$$v = \frac{D\eta}{Dt} \tag{30}$$

(notice: η here is a perturbation in the y direction this time!)
Linearized:

$$\frac{\partial \psi'}{\partial x} = \frac{\partial \eta'}{\partial t} + U\frac{\partial \eta'}{\partial x} \tag{31}$$

Insert wave solutions and apply to each side of interface:

$$\tilde{\psi}_1 = (U_1 - c)\tilde{\eta} \tag{32}$$
$$\tilde{\psi}_2 = (U_2 - c)\tilde{\eta} \tag{33}$$

Material continuity: $\tilde{\eta}$ continuous:

$$\boxed{\Delta\left[\frac{\tilde{\psi}}{U-c}\right] = 0} \tag{34}$$

4.3 Apply these matching conditions to our point jet

Solve for c: apply these solutions to our two match conditions at interface

$$-k(U_0 - c)\Phi_1 - \Phi_1 U_{1y} = k(U_0 - c)\Phi_2 - \Phi_2 U_{2y} \tag{35}$$
$$\Phi_1 = \Phi_2 \tag{36}$$

Plugging in and solving yields:

$$\boxed{c = U_0 - \frac{-U_{1y} + U_{2y}}{2k}} \tag{37}$$

This is the phase speed of *Rossby edge waves* propagating along interface $y = 0$

Note:

- for all shear values, c is purely real = no instability!
- perturbations decay away from interface
- the numerator, $-U_{1y} - (-U_{2y})$, is **directly analogous to β – a meridional change in PV** – these are truly Rossby waves!

5 Barotropic instability: piecewise-linear flow with two sharp edges (the Rayleigh problem; VallisE 8.2.1)

Now we focus on a piecewise-linear flow with *two* vorticity discontinuities. **This is the simplest model that allows for Rossby (edge) waves in two separate places in the flow!**

Note: Later on we derive necessary conditions for instability *in general*.

Rayleigh-Kuo equation governing flow stability:

$$\boxed{(U - c)(\tilde{\psi}_{yy} - k^2\tilde{\psi}) + (\beta - U_{yy})\tilde{\psi} = 0} \tag{38}$$

Simplify further:

- $\beta = 0$
- piecewise-linear flow with three regimes – and thus *two* sharp edges (**Fig 8-3**)

GROUP: Do you think there are PV gradients here? If yes, where and what is its sign?

GROUP: Which direction will Rossby waves propagate at each interface?

Basic state: three regimes (**Fig 8-3**)

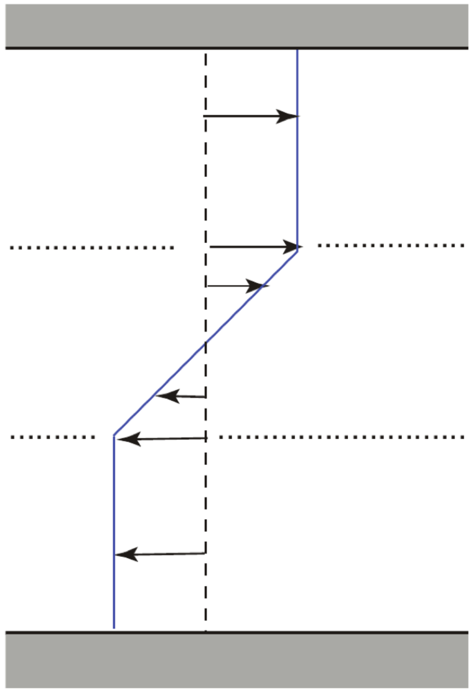

Fig. 8.3:

Figure 3: (Fig 8-3). A piecewise-linear flow with three flow regimes: constant westward flow, constant shear flow, constant eastward flow. This initial condition is *unstable* – it's breakdown is shown in Fig8-2 above!

- $y > a$: $U_1 = U_0$ (constant)
- $-a < y < a$: $U_2 = \frac{U_0}{a} y$
- $y < -a$: $U_3 = -U_0$ (constant)

Assume a **solution of the form**:

- $y > a$: $\tilde{\psi}_1 = Ae^{-k(y-a)}$
- $-a < y < a$: $\tilde{\psi}_2 = Be^{k(y-a)} + Ce^{-k(y+a)}$
- $y < -a$: $\tilde{\psi}_3 = De^{k(y+a)}$

All of these **decay away from interfaces in each region**.
As before, apply two match conditions, now at each interface
$y = a$:

$$-A[(U_0 - c)k] = B\left[(U_0 - c)k - \frac{U_0}{a}\right] - Ce^{-2ka}\left[\frac{U_0}{a} + (U_0 - c)k\right] \tag{39}$$

$$A = B + Ce^{-2ka} \tag{40}$$

$y = -a$:

$$D[(U_0 + c)k] = Be^{-2ka}\left[(U_0 + c)k + \frac{U_0}{a}\right] + C\left[\frac{U_0}{a} - (U_0 + c)k\right] \tag{41}$$

$$D = Be^{-2ka} + C \tag{42}$$

Four homogeneous equations in matrix form:

$$\begin{pmatrix} k(U_0 - c) & k(U_0 - c) - \frac{U_0}{a} & -e^{-2ka}\left[k(U_0 - c) + \frac{U_0}{a}\right] & 0 \\ 1 & -1 & -e^{-2ka} & 0 \\ 0 & -e^{-2ka}\left[k(U_0 + c) + \frac{U_0}{a}\right] & k(U_0 + c) - \frac{U_0}{a} & k(U_0 + c) \\ 0 & e^{-2ka} & 1 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

Non-trivial solutions: determinant = 0

$$\boxed{c = \left(\frac{U_0}{2ka}\right)\sqrt{(1 - 2ka)^2 - e^{-4ka}}} \tag{43}$$

Non-dimensional version: $\frac{c}{U_0}$ vs. ka

$$\boxed{\frac{c}{U_0} = \left(\frac{1}{2ka}\right)\sqrt{(1 - 2ka)^2 - e^{-4ka}}} \tag{44}$$

Instability: if $(1 - 2ka)^2 - e^{-4ka} < 0$, i.e. $ka < 0.63293$
Recall: $c = c_r + ic_i$
What is required for instability? An imaginary part of the phase speed. This occurs specifically if the quantity in the square root is negative.
How does the system evolve? (**FIG VallisBB9-5** – non-dimensional form)

- $ka > 0.63293$: $\frac{c_r}{U_0} = \left(\frac{1}{2ka}\right)\sqrt{(1 - 2ka)^2 - e^{-4ka}}$, $c_i = 0$ – pure real, **wave propagation *only***
- $ka < 0.63293$: $c_r = 0$, $\frac{c_i}{U_0} = \left(\frac{1}{2ka}\right)\sqrt{e^{-4ka} - (1 - 2ka)^2}$ – pure imaginary, **wave growth *only* – instability!**

The wave *either* propagates *or* grows – not both at the same time!
Note:

- All waves with wavenumbers $k < \frac{0.63293}{a}$ (i.e. wavelengths $\lambda = \frac{2\pi}{k} > 2\pi\frac{a}{0.63293}$) amplify exponentially
- **Maximum growth rate σ_{max} occurs for** $k = \frac{0.39}{a}$ – how would you solve for this?
- The **total flow evolution will be dominated by the *fastest-growing modes***. Example: See **FIG8-2**: this solution uses $a = 0.1 \rightarrow \lambda_{\sigma max} = 1.57 \rightarrow$ for domain of length 4, which yields wavenumber $k = \frac{4}{1.57} = 2.55$; periodic domain quantizes wavenumbers, $k = 3$ emerges – i.e. **3 waves!**

Physical insight: (**Fig Vallis BB 9.7**)

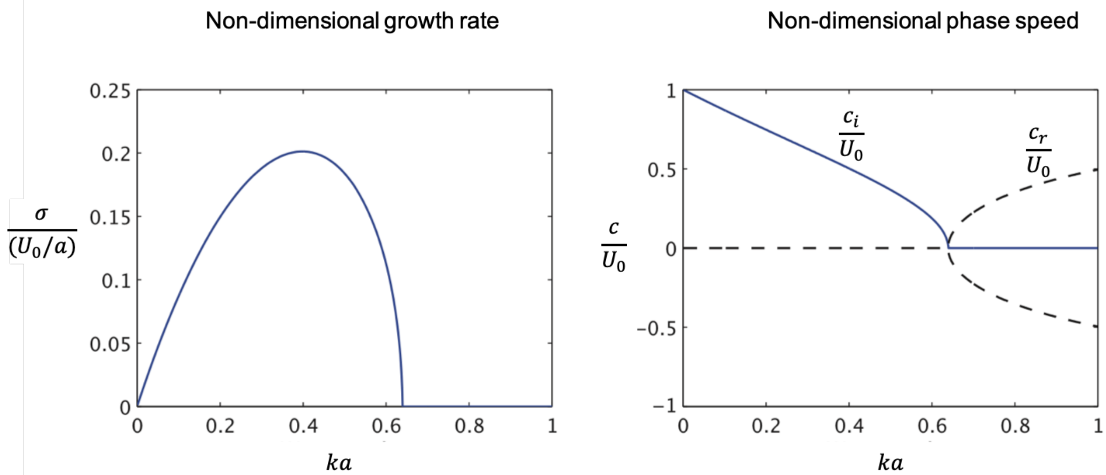

Fig. 9.5
Vallis Big Book

Figure 4: (Vallis Big Book Fig 9-5). Barotropic instability growth rate and phase speed. Note that $\sigma = c_i k$, so the appropriate non-dimensional version is $(\frac{c_i}{U_0})(ka) = \frac{\sigma}{U_0/a}$.

- This system **only has Rossby edge waves on two interfaces**
- These edge waves ***initially* propagate in opposite directions** – northern edge waves moves eastward, southern edge waves move westward.
- They then **"phase lock"** – and suddenly **stop propagating** and **start amplifying *each other*** – **GROUP: can you explain *why* using PV convservation?**
- the streamfunction (and vorticity) **perturbations tilt upshear**
- for increasing a, max growth rate occurs at fixed ka, which implies decreasing k i.e. increasing λ – **if the waves are farther apart they must be larger in order to interact**

6 Does β matter?

Necessary condition: PV gradient changes sign in the domain. (this is most important – see below for 2nd condition)

FIG8-5: Which of these zonal wind profiles *might* be unstable?

What if $\beta > 0$ – will this change your answers?

β can stabilize an unstable jet, or it can destabilize a stable jet – it depends on the profile! (Note: for sufficiently large β (how large?), the PV will become positive everywhere – i.e. stable.)

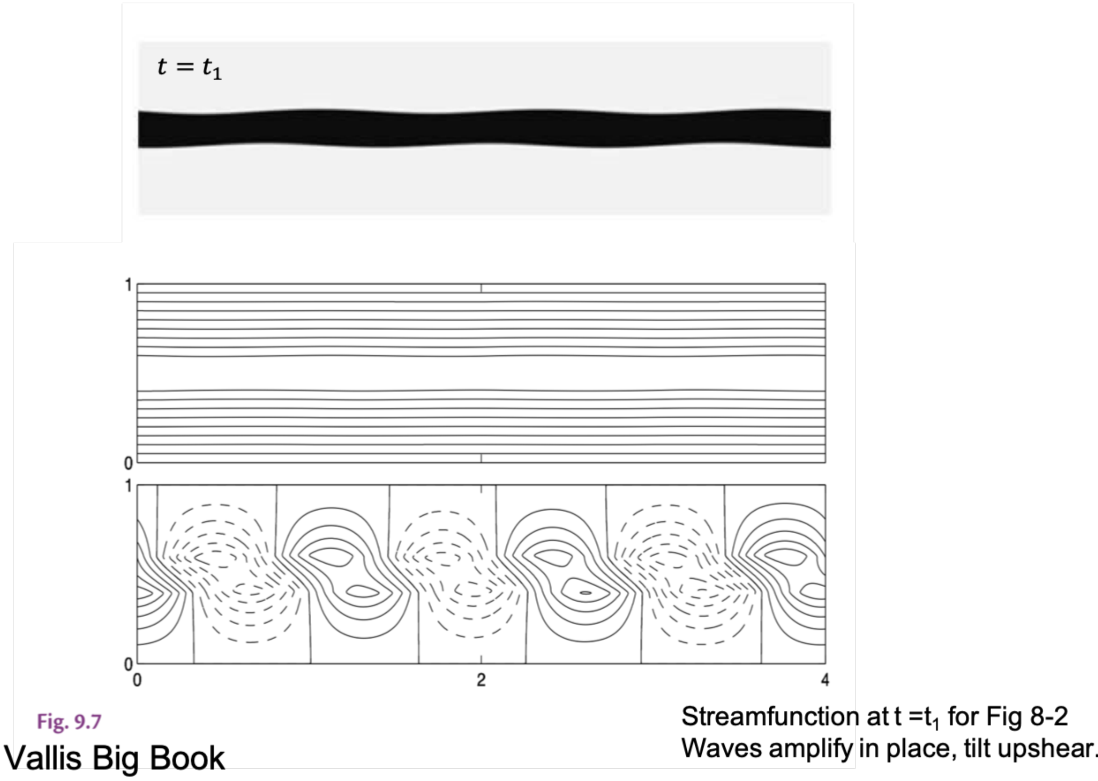

Figure 5: (Vallis Big Book Fig 9-7). What is actually happening when the instability begins.

7 (OPTIONAL) Generalization: Necessary conditions for barotropic instability

Necessary (but not sufficient) – does not guarantee instability. (= sufficient conditions for *stability*).

Two conditions:

1) Rayleigh's criterion:

Rewrite Rayleigh-Kuo as

$$\tilde{\psi}_{yy} - k^2\tilde{\psi} + \frac{\beta - U_{yy}}{U - c}\tilde{\psi} = 0 \tag{45}$$

Multiply by complex conjugate $\tilde{\psi}^*$, where

$$\tilde{\psi}\tilde{\psi}^* = \left|\tilde{\psi}\right|^2 \tag{46}$$

and integrate over domain of interest (here in y), integrating the first term by parts:

($\int_{y_1}^{y_2} \tilde{\psi}^*\tilde{\psi}_{yy}dy = \int_{y_1}^{y_2} \frac{\partial}{\partial y}\left(\tilde{\psi}^*\tilde{\psi}_y\right) - \tilde{\psi}_y^*\tilde{\psi}_y dy = \int_{y_1}^{y_2} -\tilde{\psi}_y^*\tilde{\psi}_y dy$; integral of perfect differential $= 0$ for $|y| \to \infty$)

then mult by -1 yields:

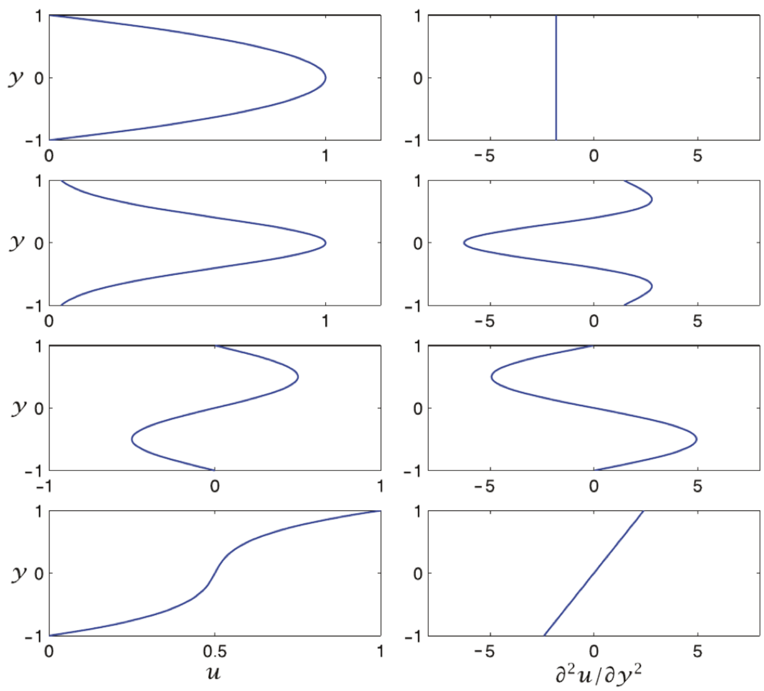

Figure 6: (Fig8-5) Zonal wind profiles. Which might be unstable?

$$\int_{y_1}^{y_2}\left(\left|\tilde{\psi}_y\right|^2+k^2\left|\tilde{\psi}\right|^2\right)dy-\int_{y_1}^{y_2}\frac{\beta-U_{yy}}{U-c}\left|\tilde{\psi}\right|^2 dy=0 \tag{47}$$

Assume $\tilde{\psi}$ vanishes at the boundaries (i.e. no perturbation for $|y| \to \infty$).

The only variable that is complex is c – only appears in second integral. Thus, first integral is purely real.

Second integral:

To get imaginary part out of denominator, multiply top and bottom of integrand by complex conjugate $(U - c^*)$, where $c^* = c_r - ic_i$

$(U-c)*(U-c^*) = U^2-Uc-Uc^*+cc^* = U^2-U(c_r+ic_i)-U(c_r-ic_i)+(c_r+ic_i)(c_r-ic_i) = U^2-Uc_r-Uc_r+(c_r^2+c_i^2) = U^2-2Uc_r+(c_r^2+c_i^2)$

Thus,

$$(U-c)(U-c^*) = |U-c|^2 \tag{48}$$

is purely real.

This yields

$$\boxed{\int_{y_1}^{y_2}(U-(c_r-ic_i))\frac{\beta-U_{yy}}{|U-c|^2}\left|\tilde{\psi}\right|^2 dy=0} \tag{49}$$

Thus, the purely imaginary component of second integral is

$$\boxed{c_i\int_{y_1}^{y_2}\frac{\beta-U_{yy}}{|U-c|^2}\left|\tilde{\psi}\right|^2 dy=0} \tag{50}$$

Two options for this to be true:

1. $c_i = 0$: stable
2. integral $= 0$

Latter occurs only if $\beta - U_{yy}$ changes sign somewhere in the domain.

1) Rayleigh's necessary condition for barotropic instability: the PV gradient ($\beta - U_{yy}$) changes sign within domain (Rayleight-Kuo inflection point criterion)

Special case: $\beta = 0$

U_{yy} changes sign in domain – when does this occur? Requires an inflection point in $U(y)$. Example: a jet!

2) Fjortoft criterion

Let's use the real part, too

$$\int_{y_1}^{y_2} (U - c_r) \frac{\beta - U_{yy}}{|U - c|^2} \left|\tilde{\psi}\right|^2 dy = \int_{y_1}^{y_2} \left(\left|\tilde{\psi}_y\right|^2 + k^2 \left|\tilde{\psi}\right|^2 \right) dy > 0 \tag{51}$$

The RHS is positive-definite.

But for instabliity, the Rayleigh-Kuo criterion says that the LHS integrand *without the factor* $(U - c_r)$ must $= 0$. Thus, the only way the integral can be > 0 is

2) Fjortoft's necessary condition for barotropic instability: $(U - U_s)(\beta - U_{yy})$ **must be positive somewhere in the domain**, where U_s is *any* real constant (typically chosen to be balue of $U(y)$ where $\beta - U_{yy} = 0$ – this is most stringent criterion; if choose U_s very large or small, then you add nothing to Rayleigh's criterion).

What does this mean? **The magnitude of the vorticity (U_y) must have extremum inside the domain (not at boundary).**