# EAPS 53600, Spring 2020
# Lec 10: Baroclinic instability (VallisE 8.4-8.7)

April 10, 2020

## 1 Introduction

2nd form of hydrodynamic instability: baroclinic instability – this creates extratropical cyclones, which creates most of the *weather* in the middle/high latitudes of rotating planets like Earth!

Occurs in fluid that has:

1. stable stratification
2. background rotation
3. **horizontal density (temperature) gradient** – i.e. **baroclinicity** (barotropic: constant density, horizontal variations in fluid height only)

**Why do these properties matter *conceptually*?**

1. Background rotation allows horizontal density gradients to exist in a balanced state (**geostrophic balance**) – you need a Coriolis force to balance horizontal pressure gradient forces.
2. Stable stratification allows for **hydrostatic balance** to hold in the vertical
3. Geostrophic balance (horizontal) + hydrostatic balance (vertical) = **thermal wind balance**
4. Thermal wind balance: horizontal density gradients = **vertical wind shear**
5. Wind shear will create opportunity for **counter-propagating Rossby waves – the essential ingredient for instability!**
6. Note: for baroclinic instability, the wind shear is in the vertical rather than horizontal direction (barotropic instability).

We now need to account for **stratification** of the fluid – a one-layer model won't cut it!

- Real world: **continuous stratification** (which you can think of as an N-layer shallow-water system, where N is a large number)
- The simplest model: **two-layer** shallow-water system – check out VallisE Ch 8.6.

Fig. 8.6:

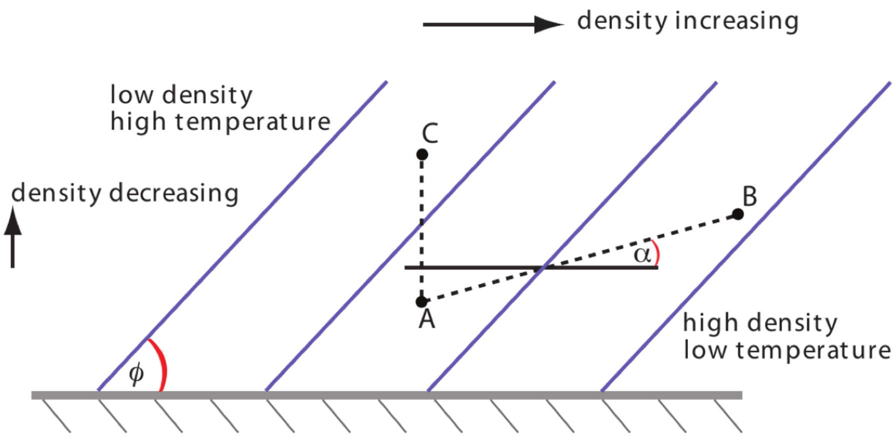


Figure 1: (Fig8-6)

## 2 A conceptual understanding: available potential energy

Instability = a fluid that is initially at rest begins to **move**. This requires **kinetic energy** – which must be drawn from the **potential energy** of the background state. Potential energy that can potentially be converted to kinetic energy is called **"available potential energy"**.

Diagram of basic state + instability: **FIG 8.6**.

- Basic state (y-z): Thermal wind balance: pressure gradient acceleration = Coriolis acceleration at all levels
- Result: isotherms (isopycnals = constant density surfaces) slope upward towards poles
- Is this basic state stable to perturbations? No, specifically for *sloped* displacements. It is possible to move a denser parcel downwards (B) and a lighter parcel upwards (A) – which would reduce the total potental energy of the system.
- General: slope $\phi > 0$ = potential energy **available** $\rightarrow$ kinetic energy

Consider:

1. Initial PE of parcels: $PE_0 = g(\rho_A z_A + \rho_B z_B)$

2. Exchange parcels A (lighter, lower) and B (denser, higher): moved denser fluid downward, lighter fluid upward

3. Final PE of parcels: $PE_f = g(\rho_A z_B + \rho_B z_A)$

$$\boxed{\Delta PE = PE_f - PE_0 = -g\Delta\rho\Delta z} \tag{1}$$

where $\Delta\rho = \rho_B - \rho_A$, $\Delta z = z_B - z_A$

If if $\rho_B > \rho_A$ and $z_B > z_A$: $\Delta PE < 0$ – The center of mass of the fluid is lowered. PE of fluid is converted into KE of perturbation, perturbation amplifies!

Define in terms of slope:

- Slope of displacement: $\alpha$
- Slope of isopycnals (isotherms): $\phi = \frac{dz}{dy}\big|_T = -\frac{\frac{\partial\rho}{\partial y}}{\frac{\partial\rho}{\partial z}}$
- Horizontal displacement distance: $L$

$\Delta PE = -g\Delta\rho\Delta z = -g\left(\frac{\partial\rho}{\partial y}L + \frac{\partial\rho}{\partial z}\alpha L\right)\alpha L$

$tan(\phi) = \frac{\Delta z}{\Delta y} = \frac{\partial\rho}{\partial y}/\frac{\partial\rho}{\partial z}$

$\frac{\partial\rho}{\partial z} = \frac{\frac{\partial\rho}{\partial y}}{tan(\phi)}$

$tan(\phi) \approx \phi$ for $\phi$ small

$\frac{\partial\rho}{\partial z} = \frac{\frac{\partial\rho}{\partial y}}{\phi}$

Thus, for $\phi$ small:

$$\boxed{\Delta PE = -gL^2\alpha\frac{\partial\rho}{\partial y}\left(1 - \frac{\alpha}{\phi}\right)} \tag{2}$$

- if $0 < \alpha < \phi$: $\Delta PE < 0$, perturbation grows
- max conversion: $\alpha = \frac{\phi}{2}$ (set $\frac{d(\Delta PE)}{d\alpha} = 0$)

5mm

**In the atmosphere, the slope of density surfaces is very shallow**: $\phi_{atm} \approx \frac{1\ km}{1000\ km} = 10^{-3}$ – but it's steep enough to provide the available potential energy required to generate extratropical cyclones!

# 3 Dynamical theory

## 3.1 System: QG equations, continuously-stratified, Boussinesq

- Interior: PV equation
- Vertical boundaries (ground, tropopause): buoyancy equation

(Note: tropopause is *not* a true rigid surface, but the high static stability of stratosphere strongly inhibits vertical motion)

**Interior flow:** QG PV conservation

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0, \qquad 0 < z < H \tag{3}$$

where QG PV in the stratified system is

$$q = \nabla^2 \psi + \beta y + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) \tag{4}$$

Boussinesq refresher:

- Density: $\rho(x, y, z) = \rho_0 + \rho'(x, y, z)$; $\rho_0$ constant; $\rho'$ only matters for *buoyancy*
- Pressure: $p(x, y, z) = p_0(z) + p'(x, y, z)$; hydrostatic background state $p_0(z) = p_{sfc} - \rho_0 g z$
- Buoyancy: $\boxed{b = -\frac{\rho'}{\rho_0} g}$

**Top/bottom boundaries:** buoyancy conservation

$$\frac{\partial b}{\partial t} + \mathbf{u} \cdot \nabla b = 0, \qquad z = 0, H \tag{5}$$

where

$$b = f_0 \frac{\partial \psi}{\partial z} \tag{6}$$

(where did this come from? Buoyancy is a perturbation from hydrostatic balance: $b = \frac{1}{\rho_0}\frac{\partial p'}{\partial z}$, $\psi = \frac{p'}{\rho_0 f_0}$, $b = \frac{1}{\rho_0}\frac{\partial(\rho_0 f_0 \psi)}{\partial z}$)

## 3.2 Basic state (using capital letters!)

- $B(y, z)$ = background meridional+vertical variation in buoyancy (i.e density, temperature)
- $U(y, z) = -\frac{\partial \overline{\psi}(y,z)}{\partial y}$ – zonal flow
- $B$ and $U$ are in **thermal wind balance** with each other: $\boxed{\frac{\partial B}{\partial y} = -f_0 \frac{\partial U}{\partial z}}$ – meridional buoyancy gradients = vertical shear of zonal wind

Recall: thermal wind balance comes from geostrophic balance (horizontal momentum) $-\frac{\partial \phi}{\partial y} = f_0 U$ and hydrostatic balance (vertical momentum) $B = \frac{\partial \phi}{\partial z}$

Let's think in terms of PV and buoyancy (both in terms of streamfunction):

- PV: $q = Q(y,z) + q'(x,y,z)$
- Buoyancy: $b = B(y,z) + b'(x,y,z)$

$$\boxed{Q(y,z) = \frac{\partial^2 \overline{\psi}}{\partial y^2} + \beta y + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \overline{\psi}}{\partial z}\right)} \tag{7}$$

(note: x-derivative term is zero here, so $\nabla^2 \overline{\psi} = \frac{\partial^2 \overline{\psi}}{\partial y^2}$)

### 3.3 Linearization about basic state

Linearize both equations about this basic state:

- $q = Q + q'$
- $b = B + b'$
- $\psi = \overline{\psi} + \psi'$
- $u = U + u'$
- $v = v'$

**Interior flow:**

$$\boxed{\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v'\frac{\partial Q}{\partial y} = 0, \qquad 0 < z < H} \tag{8}$$

where

$$\boxed{q' = \nabla^2 \psi' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right)} \tag{9}$$

and

$$\boxed{v' = \frac{\partial \psi'}{\partial x}} \tag{10}$$

**Top/bottom boundaries:**

$$\boxed{\frac{\partial b'}{\partial t} + U\frac{\partial b'}{\partial x} + v'\frac{\partial B}{\partial y} = 0 \qquad z = 0, H} \tag{11}$$

where

$$\boxed{b' = f_0\frac{\partial \psi'}{\partial z}} \tag{12}$$

and

$$\boxed{\frac{\partial B}{\partial y} = -f_0\frac{\partial U}{\partial z}} \tag{13}$$

Notice:

1. $\frac{\partial B}{\partial y}$ in the boundary equation very similar role as $\frac{\partial Q}{\partial y}$ in the interior equation. **A meridional buoyancy gradient on the top or bottom boundary is dynamically equivalent to a meridional PV gradient!**

- **meridional PV gradient → interior Rossby waves**
- **meridional buoyancy gradient → Rossby *edge* waves**

2. $\frac{\partial B}{\partial y} \to \frac{\partial U}{\partial z}$ (TWB) – wind shear between the two boundaries with Rossby edge waves.

**This setup for baroclinic instability is conceptually identical to the Rayleigh problem for barotropic instability. Wow!**

### 3.4 Normal mode (i.e. wave) solutions

Seek wave solutions to our two equations. Coefs are independent of x, not y or z. Thus, solution of form

$$\psi'(x, y, z, t) = Re\{\tilde{\psi}(y, z)e^{ik(x-ct)}\} \tag{14}$$

(similar form for $u'$, $v'$, $b'$, $q'$)
Plug in and solve: (subscripts = derivatives)

$$\boxed{(U - c)\left(\tilde{\psi}_{yy} + \left(\frac{f_0^2}{N^2}\tilde{\psi}_z\right)_z - k^2\tilde{\psi}\right) + Q_y\tilde{\psi} = 0, \qquad 0 < z < H} \tag{15}$$

$$\boxed{(U - c)\tilde{\psi}_z - U_z\tilde{\psi} = 0, \qquad z = 0, H} \tag{16}$$

These equations govern the stability of the flow – and they are **analogous to Rayleigh equations for parallel shear flow**!

**Ultimately, then, this instability involves the horizontal circulations of one Rossby wave amplifying another Rossby wave (with opposite-signed PV gradient), and vice versa. Physically, then, what is the difference?**

- **Barotropic** instability: the circulations interact in the same horizontal plane (e.g. one to the north and one to the south). If they are close enough together horizontally, they can mutually amplify.

- **Baroclinic** instability: the circulations interact ***vertically*** – their horizontal circulations project downward (from above) or upward (from below) over some depth of the fluid. If they are close enough together *vertically*, they can mutually amplify.

Let's see how this works mathematically.

# 4 Simplest model: Eady problem (V8.5)

Two mathematical descriptions of baroclinic instability:

1. **Eady problem**: f-plane ($\beta = 0$), two Rossby edge waves; simplest possible model

2. **Charney problem**: $\beta > 0$, interior Rossby wave + Rossby edge wave; mathematically more complex but conceptually identical (not covered in this book)

**Here we do Eady.**
Simplifying assumptions:

1. f-plane $f = f_0$
2. $N^2$ constant (uniformly stratified)
3. Top/bottom boundaries: rigid lids ($z = 0$ ground, $z = H$ tropopause)
4. Basic state flow: uniform shear, $\boxed{U_0(z) = \Lambda z}$, $\Lambda = \frac{U}{H}$, $U = U_0(z = H)$

Assumptions more valid for atmosphere, but qualitative outcome applies to ocean too.
**Basic state**:
1) Streamfunction: ($U = -\frac{\partial\overline{\Psi}}{\partial y}$)

$$\boxed{\overline{\Psi} = -\Lambda z y} \tag{17}$$

2) Buoyancy: varies only in $y$ ($B = f_0 \frac{\partial\overline{\psi}}{\partial z}$)

$$\boxed{B = -f_0 \Lambda y} \tag{18}$$

3) PV: $= 0$ ($\overline{\Psi}$ is linear in $y$ and $z$)

$$\boxed{Q = \frac{\partial^2\overline{\psi}}{\partial y^2} + \frac{H^2}{L_d^2}\frac{\partial^2\overline{\psi}}{\partial z^2} = \mathbf{0}} \tag{19}$$

where $L_d = \frac{NH}{f_0}$

**3D DIAGRAM OF BASIC STATE BAROCLINIC from comparison with barotropic**

Can write our PV as: $Q = -\frac{\partial^2 U}{\partial y^2} + \frac{f_0}{N^2}\frac{\partial B}{\partial z}$ (first term = 0)
$B = \textit{constant}$ outside of interior, decreasing in $y$ direction on boundaries. Thus:

- $z = H$: (from neg to zero moving upward) $\frac{\partial}{\partial z}(B_y) > 0 \rightarrow \frac{\partial Q}{\partial y} > 0$ – westward RW propagation
- $z = 0$: (from zero to neg moving upward) $\frac{\partial}{\partial z}(B_y) < 0 \rightarrow \frac{\partial Q}{\partial y} < 0$ – eastward RW propagation

**This should feel familiar: it's exactly the same as the Rayleigh problem – piecewise linear flow for barotropic instability. In both cases: PV gradients of opposite signs confined to two interfaces.**

**Conceptually, what system have we created?** No interior PV gradient, only meridional buoyancy gradients at top and bottom – i.e. **only counter-propagating Rossby edge waves at the boundaries, none in interior!** We have what we need for instability...

## 4.1 Linearization

### 4.1.1 Interior

Recall linearized PV equation:

$$\frac{\partial q'}{\partial t} + U_0 \frac{\partial q'}{\partial x} + v' \frac{\partial Q}{\partial y} = 0, \qquad 0 < z < H \tag{20}$$

now $q' = \nabla^2 \psi' + \frac{H^2}{L_d^2} \frac{\partial^2 \psi'}{\partial z^2}$ and $v' = \frac{\partial \psi'}{\partial x}$
Plug in to put in terms of $\psi'$:
Thus, linearized PV equation in terms of $\psi'$ is:

$$\boxed{\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right)\left(\nabla^2 \psi' + \frac{H^2}{L_d^2} \frac{\partial^2 \psi'}{\partial z^2}\right) = 0, \qquad 0 < z < H} \tag{21}$$

### 4.1.2 Boundary conditions

Vertical boundary conditions:

- $w = 0$ at $z = 0$
- $w = 0$ at $z = H$

Lateral boundary conditions: could do doubly-periodic (i.e. wave solutions in $x$ and $y$) or channel ($y$ only).

Here we do **zonal channel** of meridional width $L$. Hence, all wave perturbations should decay to zero at the north/south boundaries, i.e.:

- $\psi = 0$ at $y = +\frac{L}{2}$ (no motion)
- $\psi = 0$ at $y = -\frac{L}{2}$

Recall linearized buoyancy equation:

$$\frac{\partial b'}{\partial t} + U_0 \frac{\partial b'}{\partial x} + v' \frac{\partial B}{\partial y} = 0 \qquad z = 0, H \tag{22}$$

where $b' = f_0 \frac{\partial \psi'}{\partial z}$ and $\frac{\partial B}{\partial y} = -f_0 \frac{\partial U}{\partial z} = -f_0 \Lambda$
Plug in to put in terms of $\psi'$:

$$\boxed{\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right) \frac{\partial \psi'}{\partial z} - \Lambda \frac{\partial \psi'}{\partial x} = 0 \qquad , z = 0, H} \tag{23}$$

Channel walls require **modal solutions in** $y$, i.e.

$$\boxed{\psi'(x, y, z, t) = Re\{\Phi(z) sin(ly) e^{ik(x-ct)}\}} \tag{24}$$

with meridional wavenumber $l = \frac{n\pi}{L}$ and $n$ is a positive integer
Plug in:

1) Interior (PV):

$$\boxed{(\Lambda z - c)\left[\frac{H^2}{L_d^2}\frac{\partial^2\Phi}{\partial z^2} - (k^2+l^2)\Phi\right] = 0} \tag{25}$$

2) $z = 0$:

$$\boxed{c\frac{d\Phi}{\partial z} - \Lambda\Phi = 0} \tag{26}$$

3) $z = H$:

$$\boxed{(c - \Lambda H)\frac{d\Phi}{\partial z} + \Lambda\Phi = 0} \tag{27}$$

These are simply Eqns (15) applied specifically to Eady problem.
For $\Lambda z \neq c$, interior equation can be written:

$$\left[H^2\frac{\partial^2\Phi}{\partial z^2} - \mu^2\Phi\right] = 0 \tag{28}$$

where

$$\mu^2 = L_d^2(k^2+l^2) \tag{29}$$

$\mu$ **is horizontal wavenumber rescaled by the deformation radius**

Solution is

$$\boxed{\Phi(z) = Acosh(\mu\hat{z}) + Bsinh(\mu\hat{z})} \tag{30}$$

where

$$\hat{z} = \frac{z}{H} \tag{31}$$

Thus $\mu$ **determines *vertical* structure of solution – smaller** $\mu$ **(i.e. larger** $\lambda$**)** $\rightarrow$ **deeper penetration – a larger disturbance *also* penetrates deeper up/down into the fluid!**)

Plug into boundary conditions:

$$A[\Lambda H] + B[\mu c] = 0 \tag{32}$$

$$A[(c-\Lambda H)\mu sinh(\mu) + \Lambda H cosh(\mu)] + B[(c-\Lambda H)\mu cosh(\mu) + \Lambda H sinh(\mu)] = 0 \tag{33}$$

As usual, non-trivial solutions only if determinant of coefficient matrix = 0, i.e.:

$$c^2 - Uc + U^2\left(\frac{1}{\mu}coth(\mu) - \frac{1}{\mu^2}\right) = 0 \tag{34}$$

where $U = \Lambda H$; recall: $coth(\mu) = \frac{cosh(\mu)}{sinh(\mu)}$

Solve for $c$:

$$\boxed{c = \frac{U}{2} \pm \frac{U}{\mu}\sqrt{\left(\frac{\mu}{2} - coth(\frac{\mu}{2})\right)\left(\frac{\mu}{2} - tanh(\frac{\mu}{2})\right)}} \tag{35}$$

**FIG8.7**

Growth/decay: $c_i \neq 0$ – negative under square root
$tanh(\frac{\mu}{2}) < \frac{\mu}{2}$ for all $\mu \rightarrow$ second term positive
Thus, requires $\frac{\mu}{2} < coth(\frac{\mu}{2})$. This occurs when

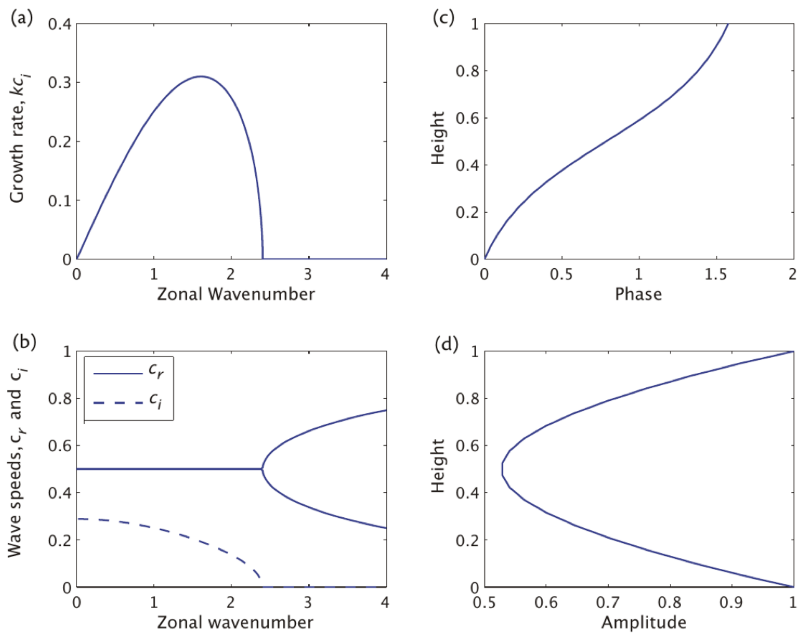


Fig. 8.7:

Figure 2: (Fig8-7)

$$\boxed{\mu < \mu_c = 2.4} \tag{36}$$

with growth rate is

$$\boxed{\sigma = kc_i = k\frac{U}{\mu}\sqrt{\left(\frac{\mu}{2} - coth(\frac{\mu}{2})\right)\left(\frac{\mu}{2} - tanh(\frac{\mu}{2})\right)}} \tag{37}$$

The maximum growth rate occurs at

$$\boxed{\mu_{max} = 1.61} \tag{38}$$

Most unstable $\mu$ occurs has $n = 1$ (i.e. $l = \frac{\pi}{L}$)

Growth rates are larger for $l^2 \ll k^2$ – i.e. large meridional scale of waves. For this case, **the maximum growth rate is called *Eady growth rate***, and is:

$$\boxed{\sigma_E = \frac{0.31U}{L_d} = \frac{0.31\Lambda H}{L_d} = \frac{0.31\Lambda f}{N}} \tag{39}$$

For these growing modes, phase speed is

$$\boxed{c_r = 0.5U} \tag{40}$$

(i.e. the average zonal speed in the layer)

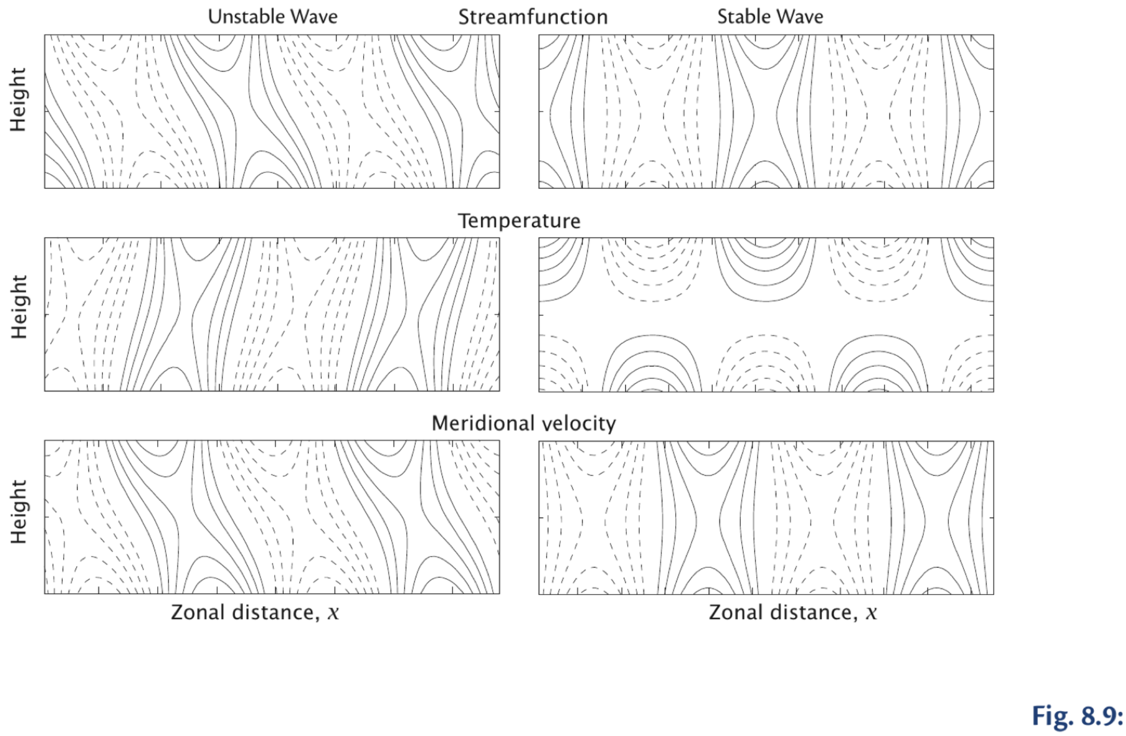


Figure 3: (Fig8-9)

Translate to physical space: for $l$ small, $\mu = L_d k$. Thus instability occurs for:

$$\boxed{k < \frac{2.4}{L_d}} \tag{41}$$

i.e. ($\lambda = \frac{2\pi}{k}$)

$$\boxed{\lambda > 2.6 L_d} \tag{42}$$

Thus, there exists **a shortwave cutoff for baroclinic instability**; this cutoff scales with the deformation radius.

Maximum instability occurs for:

$$\boxed{k_{max} = \frac{1.6}{L_d}} \tag{43}$$

i.e.

$$\boxed{\lambda_{max} = 3.9 L_d} \tag{44}$$

**FIG 8.9**

Requirement: phase-locking of counter propagating rossby edge waves.

# 5 Comparing barotropic and baroclinic instability (Rayleigh vs. Eady)

See PDF hand-written notes

### 5.1 Mechanistic understanding in the lab

You created baroclinic instability in the first rotating tank lab at the highest rotation rate!

### 5.2 Mechanistic understanding in real world

Movie

## 6 Necessary conditions for baroclinic instability

Same procedure as for barotropic instability: Multiply eqns by $\tilde{\psi}^*$ and integrate over domain in $y$ and $z$. (See VallisE 8.7)

$$\int_0^H \int_{y_1}^{y_2} \left[ \left|\tilde{\psi}_y\right|^2 + \frac{f_0^2}{N^2} \left|\psi_z\right|^2 + k^2 \left|\psi\right|^2 \right] dydz - \int_{y_1}^{y_2} \left( \int_0^H \frac{Q_y}{U-c} \left|\tilde{\psi}\right|^2 dz + \left[ \frac{\frac{f_0^2}{N^2} U_z \left|\tilde{\psi}\right|^2}{U-c} \right]_0^H \right) dy = 0 \tag{45}$$

- First term: pure real ($c_i = 0$)
- Second term: complex

Growth/decay: $c_i \neq 0$
Imaginary component of second term:

$$-c_i \int_{y_1}^{y_2} \left( \int_0^H \frac{Q_y}{|U-c|^2} \left|\tilde{\psi}\right|^2 dz + \left[ \frac{\frac{f_0^2}{N^2} U_z \left|\tilde{\psi}\right|^2}{|U-c|^2} \right]_H - \left[ \frac{\frac{f_0^2}{N^2} U_z \left|\tilde{\psi}\right|^2}{|U-c|^2} \right]_0 \right) dy = 0 \tag{46}$$

Requires either:

- $c_i = 0$ – no growth/decay
- $c_i \neq 0$, rest of LHS $= 0$

**Charney-Stern-Pedlosky (CSP) necessary condition for baroclinic instability**: **ONE** of the following criteria must be satisfied:

1. $Q_y$ changes sign in the interior
2. $Q_y$ is opposite sign to $U_z$ at upper boundary ($z = H$)
3. $Q_y$ is same sign to $U_z$ at lower boundary ($z = 0$)
4. If $Q_y = 0$, $U_z$ is same sign at upper and lower boundaries

Application to Earth's atmosphere:

- $Q_y$: *typically* dominated by $\beta$, which is positive everywhere – eliminates (1) and (4)
- $U_z$: also *typically* positive everywhere (westerly jet streams aloft, surface easterlies) – eliminates (2)
- $\boxed{\text{(3) is most common: } Q_y > 0 \text{ and } U_z(0) > 0}$

Thus, our atmosphere, particularly at mid-latitudes in the vicinity of the jet stream, is potentially baroclinically unstable.