Stability Analysis of Networked Control Systems with Unknown Inputs

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Abstract— **Unknown Input Observers (UIO) use the known plant's inputs and outputs to generate state estimates for plants with unknown inputs. In many cases, the UIO's inputs are transmitted through a communication network, which is a key component in modern Networked Control Systems (NCS). Most of the developed UIOs in the literature are designed for non-networked systems. In this paper, the objective is to study the effect of unknown inputs and network perturbations on state observation and stability using UIOs. Given an UIO design for any non-networked system, we derive the dynamics of the UIO-based NCS, also referred to as Networked Unknown Input Observer (N_{et}UIO). The stability of the N_{et}UIO is analyzed by deriving a stability-guaranteeing bound on the networked-induced perturbation. Numerical simulations are provided to highlight the applicability of the obtained bounds.**

I. Introduction and Literature Review

The research presented in this paper aims to analyze the network perturbation and unknown input disturbance effects on the plant state estimation system stability, given that a Networked Control System (NCS) framework is considered. NCSs are used to control, monitor, and manage Cyber Physical Systems (CPS) such as smart power-grids, transportation and telecommunication networks, and other intelligent dynamical systems. Data exchanged within these systems (i.e., between sensors, actuators, state estimators or controllers) are often transmitted through digital communication networks. Estimators in general, and observers in specific, use the known plant's inputs and outputs to generate estimates for the state of the plant. The closed loop system is then monitored through a controller that often use the estimated state of the plant to generate control commands.

For example, state-estimators and observers are used in power networks to precisely estimate the plant state (i.e., the bus voltages and phase angles), which is crucial for successful control and operation for the modern smart-grid. The generated real-time dynamic estimates of the bus voltages and angles facilitate calculating optimal power flows for transmission lines [1]. One of the main objectives behind utilizing state estimators and observers for dynamical systems is to augment or replace expensive sensors in control systems [2]. For that and various reasons, the analysis and design of dynamic robust observers for linear and nonlinear systems, for systems with known and unknown inputs and disturbances have received noticeable attention in the past few decades.

Luenberger was the first to propose, analyze and design observers [3]–[5]. The well-known Luenberger Observer is still utilized for various engineering applications. Furthermore, observers for systems with unknown inputs and disturbances, also called Unknown Input Observers (UIO), have been extensively studied since the late seventies. The following are some well known research efforts on UIOs: Bhattacharyya [6], Chang *et al.* [7], Chen *et al.* [8], Chen and Patton [9], Corless and Tu [10], Darouach *et al.* [11], Hui and Żak [12]. For more references on different UIO architectures, we direct the reader to [2, p. 431].

II. Research Gaps and Paper Overview

As mentioned in Section I, observers use the known plant's inputs and outputs to generate estimates for the state of the plant. The closed loop system is then controlled through a controller that often use the estimated plant state to generate control commands. Observers in different decentralized large-scale systems such as transportation networks, power plants, and remotely controlled mobile agents are often distributed. Hence, the UIO's input (i.e., the plant's input and output) is transmitted through a communication network, which is a key component in modern NCSs. Hence, most observers for systems with unknown inputs are Networked Unknown Input Observers (N_{et}UIO).

Most of the developed UIOs in the literature are designed for non-networked systems as in [6]–[12]. The UIO's input is assumed to be transmitted without any disturbances, perturbations, or time-delays. Since communication networks are inserted in the feedback loops of most decentralized control systems, the analysis and design of the observers for *networked* systems with unknown inputs becomes a necessity. The network effect can be either modeled as a perturbation or time-delay to the transmitted signals. In [14] and [15], we analyzed the effect of perturbation of the signals exchanged through communication networks for decentralized observer-based control for systems with only known inputs. In our most recent work, we applied a time-delay analysis for NCSs with applications to power networks, for systems with only known inputs as in [16] and for systems with unknown inputs as in [17]. In this

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paper, the proposed design targets the signals perturbation representations of a network. In Section III, we review an Unknown Input Observer (UIO) design for non-networked system from the UIO literature as in [2] and we present the problem formulation. The dynamics of the system given a perturbation model of the network is formulated in Section IV and the stability bounds for that model are derived in Sections V and VI. Section VII includes numerical examples to illustrate the usefulness and applicability of the proposed model. Conclusions are presented in Section VIII.

III. System Modeling and Problem Formulation

As mentioned in the introduction, the research presented in this paper aims to analyze the network perturbation and unknown input disturbance effects to the plant state estimation, given that a Networked Control System (NCS) framework is considered that utilizes an UIO architecture. The non-networked UIO architecture used in this paper is presented in [2], but the framework proposed in this paper is valid for any linear UIO architecture. The input to the UIO block is the delayed or perturbed version of the plant's output ($\boldsymbol{y}$), that is $\hat{\boldsymbol{y}}$, and the input $\boldsymbol{u}_1$ to the plant. These two quantities are assumed to be known to the UIO/Controller block. The unknown input for the plant is $\boldsymbol{u}_2$ (unknown plant disturbances, nonlinearities or actuator faults). The UIO & Controller block computes the estimated state of the plant, given that the plant is subject to unknown inputs and given that a network is inserted in the feedback loops of the control system. A control action is then computed by the controller based on the estimate of the plant state.

A. *Observer Design Review for Non-Networked Systems with Unknown Inputs [2]*

The observer design and state estimations for non-networked systems with unknown inputs used in this paper is based on a projector operator approach used in [2]. In this paper, we assume a Linear Time-Invariant (LTI) class of systems. The modeled plant can be a linearized representation of a nonlinear plant.

The linearized plant dynamics can be written as:

$$\dot{\boldsymbol{x}}_p = \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_p^{(1)}\boldsymbol{u}_1 + \boldsymbol{B}_p^{(2)}\boldsymbol{u}_2, \tag{1}$$

$$\boldsymbol{y} = \boldsymbol{C}_p\boldsymbol{x}_p \tag{2}$$

where $\boldsymbol{A}_p, \boldsymbol{B}_p^{(1)}, \boldsymbol{B}_p^{(2)}$ and $\boldsymbol{C}_p$ are all known system parameters. The dynamics of the UIO presented in [2] for non-networked systems are as follows:

$$\dot{\boldsymbol{x}}_c = \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_c^{(1)}\boldsymbol{y} + \boldsymbol{B}_c^{(2)}\boldsymbol{u}_1,$$

where

$$\boldsymbol{A}_c = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}_p)(\boldsymbol{A}_p - \boldsymbol{L}\boldsymbol{C}_p), \boldsymbol{B}_c^{(2)} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}_p)\boldsymbol{B}_P^{(1)}$$
$$\boldsymbol{B}_c^{(1)} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}_p)(\boldsymbol{A}_p\boldsymbol{M} + \boldsymbol{L} - \boldsymbol{L}\boldsymbol{C}_p\boldsymbol{M}),$$

$\boldsymbol{M} \in \mathbb{R}^{n\times p}$ is chosen such that $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C}_p)\boldsymbol{B}_p^{(2)} = \boldsymbol{O}$ and $\boldsymbol{L}$ is an added gain to improve the convergence of the estimated state ($\hat{\boldsymbol{x}}_p$). The addition of the communication network perturbs the UIO's inputs (which are $\boldsymbol{y}$ and $\boldsymbol{u}_1$), as the observer uses the plant's input and output to estimate the state of the plant. Hence, the dynamics of the UIO are as follows:

$$\dot{\boldsymbol{x}}_c = \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_c^{(1)}\hat{\boldsymbol{y}} + \boldsymbol{B}_c^{(2)}\hat{\boldsymbol{u}}_1, \tag{3}$$

$$\hat{\boldsymbol{x}}_p = \boldsymbol{x}_c + \boldsymbol{M}\hat{\boldsymbol{y}}. \tag{4}$$

In this section, we assume that state-feedback control is used:

$$\boldsymbol{u}_1 = -\boldsymbol{K}\hat{\boldsymbol{x}}_p = -\boldsymbol{K}\boldsymbol{x}_c - \boldsymbol{K}\boldsymbol{M}\hat{\boldsymbol{y}}. \tag{5}$$

We can write the dynamics of the control as follows:

$$\begin{cases} \dot{\boldsymbol{x}}_c = \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_c^{(1)}\hat{\boldsymbol{y}} + \boldsymbol{B}_c^{(2)}\hat{\boldsymbol{u}}_1 \\ \boldsymbol{u}_1 = \boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_c\hat{\boldsymbol{y}}, \end{cases} \tag{6}$$

where $\boldsymbol{C}_c = -\boldsymbol{K}$ and $\boldsymbol{D}_c = -\boldsymbol{K}\boldsymbol{M}$.

IV. Dynamics of the Closed-Loop System

In this section, and in various papers from the NCS literature, the network effect can be modeled as a perturbation:

$$\hat{\boldsymbol{y}} = \boldsymbol{y} - \boldsymbol{e}_y, \;\; \hat{\boldsymbol{u}}_1 = \boldsymbol{u}_1 - \boldsymbol{e}_{u_1},$$

where $\boldsymbol{e}$ is the networked induced error. The objective of this section, is to formulate the global dynamics of the overall closed loop states of the system, namely $\boldsymbol{x}_p$, $\boldsymbol{x}_c$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_{u_1}$ where the dynamics of the non-networked UIO are previously derived (i.e., we assume that the developments in this section assume that matrices $\boldsymbol{A}_c, \boldsymbol{B}_c^{(1)}, \boldsymbol{B}_c^{(2)}, \boldsymbol{C}_c$ and $\boldsymbol{D}_c$ are all computed for the non-networked system as in any UIO architecture from the literature). The plant dynamics can be written as follows,

$$\begin{aligned} \dot{\boldsymbol{x}}_p &= \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_p^{(1)}\hat{\boldsymbol{u}}_1 + \boldsymbol{B}_p^{(2)}\boldsymbol{u}_2 \\ &= \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_p^{(1)}(\boldsymbol{u}_1 - \boldsymbol{e}_{u_1}) + \boldsymbol{B}_p^{(2)}\boldsymbol{u}_2 \\ &= \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_p^{(1)}\left(\boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_c\left(\boldsymbol{y} - \boldsymbol{e}_y\right) - \boldsymbol{e}_{u_1}\right) \\ &\quad + \boldsymbol{B}_p^{(2)}\boldsymbol{u}_2 \\ &= \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_p^{(1)}\left(\boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_c\left(\boldsymbol{C}_p\boldsymbol{x}_p - \boldsymbol{e}_y\right) - \boldsymbol{e}_{u_1}\right) \\ &\quad + \boldsymbol{B}_p^{(2)}\boldsymbol{u}_2 \\ \dot{\boldsymbol{x}}_p &= \left(\boldsymbol{A}_p + \boldsymbol{B}_p^{(1)}\boldsymbol{D}_c\boldsymbol{C}_p\right)\boldsymbol{x}_p + \boldsymbol{B}_p^{(1)}\boldsymbol{C}_c\boldsymbol{x}_c - \boldsymbol{B}_p^{(1)}\boldsymbol{D}_c\boldsymbol{e}_y \\ &\quad - \boldsymbol{B}_p^{(1)}\boldsymbol{e}_{u_1} + \boldsymbol{B}_p^{(2)}\boldsymbol{u}_2. \end{aligned} \tag{7}$$

Similarly, the controller dynamics can be written as,

$$\begin{aligned} \dot{\boldsymbol{x}}_c &= \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_c^{(1)}\hat{\boldsymbol{y}} + \boldsymbol{B}_c^{(2)}\hat{\boldsymbol{u}}_1 \\ &= \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_c^{(1)}\left(\boldsymbol{C}_p\boldsymbol{x}_p - \boldsymbol{e}_y\right) \\ &\quad + \boldsymbol{B}_c^{(2)}\left(\boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_c\left(\boldsymbol{y} - \boldsymbol{e}_y\right) - \boldsymbol{e}_{u_1}\right) \\ &= \left(\boldsymbol{B}_c^{(1)}\boldsymbol{C}_p + \boldsymbol{B}_c^{(2)}\boldsymbol{D}_c\boldsymbol{C}_p\right)\boldsymbol{x}_p + \left(\boldsymbol{A}_c + \boldsymbol{B}_c^{(2)}\boldsymbol{C}_c\right)\boldsymbol{x}_c \\ &\quad - \left(\boldsymbol{B}_c^{(1)} + \boldsymbol{B}_c^{(2)}\boldsymbol{D}_c\right)\boldsymbol{e}_y - \boldsymbol{B}_c^{(2)}\boldsymbol{e}_{u_1} \end{aligned} \tag{8}$$

Let $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{x}_p^\top & \boldsymbol{x}_c^\top\end{bmatrix}^\top$ and $\boldsymbol{e} = \begin{bmatrix}\boldsymbol{e}_y^\top & \boldsymbol{e}_{u_1}^\top\end{bmatrix}^\top$. We can now write equations (7–8) in a matrix-vector form as follows:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\Lambda_1}\boldsymbol{x}(t) + \boldsymbol{\Lambda_2}\boldsymbol{e}(t) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t), \tag{9}$$

where

$$\boldsymbol{\Lambda}_1 = \begin{bmatrix} \boldsymbol{A}_p + \boldsymbol{B}_p^{(1)}\boldsymbol{D}_c\boldsymbol{C}_p & \boldsymbol{B}_p^{(1)}\boldsymbol{C}_c \\ \boldsymbol{B}_c^{(1)}\boldsymbol{C}_p + \boldsymbol{B}_c^{(2)}\boldsymbol{D}_c\boldsymbol{C}_p & \boldsymbol{A}_c + \boldsymbol{B}_c^{(2)}\boldsymbol{C}_c \end{bmatrix},$$

$$\boldsymbol{\Lambda}_2 = \begin{bmatrix} -\boldsymbol{B}_p^{(1)}\boldsymbol{D}_c & -\boldsymbol{B}_p^{(1)} \\ -\left(\boldsymbol{B}_c^{(1)} + \boldsymbol{B}_c^{(2)}\boldsymbol{D}_c\right) & -\boldsymbol{B}_c^{(2)} \end{bmatrix}, \ \boldsymbol{\Omega_1} = \begin{bmatrix} \boldsymbol{B}_p^{(2)} \\ \boldsymbol{O} \end{bmatrix},$$

$\boldsymbol{\Lambda_2 e}(t)$ is the perturbation due to the network, and $\boldsymbol{\Omega_1 u}_2(t)$ is the perturbation due to the unknown input. By the design of the non-networked UIO, $\boldsymbol{\Lambda}_1$ is asymptotically stable matrix with strictly negative eigenvalues.

V. Stability Analysis and Perturbation Bounds for the N$_{\text{ET}}$UIO

In this section, we derive bounds on the networked-induced perturbations, such that the stability of the UIO-based NCS is guaranteed. The behavior of the network-induced error ($\boldsymbol{e}(t)$), is mainly determined by the architecture of the NCS and the scheduling protocol. Hence, we need to derive an error quantity that relates to the NCS states, namely $\boldsymbol{e}_x(t)$. This methodology is standard in the NCS literature (i.e., see [24]). Consider the time interval between transmissions: $t \in [t_i, t_{i+1}]$ where $i = 0, 1, 2, \ldots$, we get:

$$\hat{\boldsymbol{y}}(t) = \boldsymbol{y}(t_i) = \boldsymbol{C}_p\boldsymbol{x}_p(t_i)$$

and

$$\hat{\boldsymbol{u}}_1(t) = \boldsymbol{u}_1(t_i) = \boldsymbol{C}_c\boldsymbol{x}_c(t_i) + \boldsymbol{D}_c y(t_i) = \boldsymbol{C}_c\boldsymbol{x}_c(t_i) + \boldsymbol{D}_c\boldsymbol{C}_p\boldsymbol{x}_p(t_i).$$

Recall that

$$\boldsymbol{e}_y(t) = \boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t) = \boldsymbol{C}_p\boldsymbol{x}_p(t) - \boldsymbol{C}_p\boldsymbol{x}(t_i) = \boldsymbol{C}_p(\boldsymbol{x}_p(t) - \boldsymbol{x}(t_i))$$

and

$$\begin{aligned} \boldsymbol{e}_{u_1} &= \boldsymbol{u}_1(t) - \hat{\boldsymbol{u}}_1(t) = \boldsymbol{C}_c\boldsymbol{x}_c(t) + \boldsymbol{D}_c\boldsymbol{C}_p\boldsymbol{x}_p(t) \\ &\quad - (\boldsymbol{C}_c\boldsymbol{x}_c(t_i) + \boldsymbol{D}_c\boldsymbol{C}_p\boldsymbol{x}_p(t_i)) \\ &= \boldsymbol{C}_c(\boldsymbol{x}_c(t) - \boldsymbol{x}_c(t_i)) + \boldsymbol{D}_c\boldsymbol{C}_p(\boldsymbol{x}_p(t) - \boldsymbol{x}_p(t_i)). \end{aligned}$$

Let $\boldsymbol{e}_x(t) = \boldsymbol{x}(t) - \boldsymbol{x}(t_i)$, then as in [24] we can write

$$\boldsymbol{\Lambda_2 e}(t) = \boldsymbol{\Lambda_2} \begin{bmatrix} \boldsymbol{C}_p & \boldsymbol{O} \\ \boldsymbol{D}_c\boldsymbol{C}_p & \boldsymbol{C}_c \end{bmatrix} \boldsymbol{e}_x(t) = \boldsymbol{E}\boldsymbol{e}_x(t).$$

The dynamics of the perturbed system can now be written as,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\Lambda_1 x}(t) + \boldsymbol{E}\boldsymbol{e}_x(t) + \boldsymbol{\Omega_1 u}_2(t). \tag{10}$$

Theorem 1. *For the perturbed N$_{et}$UIO as represented in (10) and for a Hurwitz $\boldsymbol{\Lambda}_1$, we have $\boldsymbol{P} = \boldsymbol{P}^\top \succ \boldsymbol{O}$, is the solution to the Lyapunov matrix equation*

$$\boldsymbol{\Lambda}_1^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{\Lambda}_1 = -2\boldsymbol{Q},$$

for a given $\boldsymbol{Q} = \boldsymbol{Q}^\top \succ \boldsymbol{O}$. If $\|\boldsymbol{u}_2(t)\| < \mu_x\|\boldsymbol{x}(t)\|$, where $\mu_x > 0$, and if $\lambda_{min}(\boldsymbol{Q}) - \mu_x\lambda_{max}(\boldsymbol{P})\|\boldsymbol{\Omega}_1\| > 0$ and if the norm of the network induced perturbation $\|\boldsymbol{e}_x\|$ satisfies $\|\boldsymbol{e}_x\| < \zeta\|\boldsymbol{x}\|$ where

$$\zeta \leq \frac{\lambda_{min}(\boldsymbol{Q}) - \mu_x\|\boldsymbol{\Omega_1}\|\lambda_{max}(\boldsymbol{P})}{\lambda_{max}(\boldsymbol{P})\|\boldsymbol{E}\|},$$

then the origin is a globally exponentially stable equilibrium point of the perturbed N$_{et}$UIO (10).

Proof. Since $\boldsymbol{\Lambda}_1$ is asymptotically stable, the solution $\boldsymbol{P}$ to the Lyapunov matrix equation

$$\boldsymbol{\Lambda}_1^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{\Lambda}_1 = -2\boldsymbol{Q},$$

is symmetric positive definite (i.e., $\boldsymbol{P} = \boldsymbol{P}^\top \succ \boldsymbol{O}$.) for a given $\boldsymbol{Q} = \boldsymbol{Q}^\top \succ \boldsymbol{O}$.

Let $V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{x}$ be a Lyapunov candidate function, used to verify stability and establish bounds on the network-induced perturbation. Particularly, we need to find a bound on the perturbation $\|\boldsymbol{e}_x\|$ such that $\dot{V}(\boldsymbol{x}) < 0$. Taking the derivative of the Lyapunov candidate function, we get,

$$\dot{V}(\boldsymbol{x}) = \boldsymbol{x}^\top\boldsymbol{P}\dot{\boldsymbol{x}} \tag{11}$$

Substituting (10) into (11), we get:

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= \boldsymbol{x}^\top\boldsymbol{P}\left(\boldsymbol{\Lambda}_1\boldsymbol{x} + \boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{\Omega}_1\boldsymbol{u}_2\right) \\ &= \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Lambda}_1\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Lambda}_1\boldsymbol{x} \\ &\quad + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Omega}_1\boldsymbol{u}_2 \\ &= \frac{1}{2}\boldsymbol{x}^\top\left(\boldsymbol{P}\boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_1^\top\boldsymbol{P}\right)\boldsymbol{x} + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Omega}_1\boldsymbol{u}_2 \\ &= -\boldsymbol{x}^\top\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Omega}_1\boldsymbol{u}_2 \end{aligned} \tag{12}$$

Applying the assumed bound on the unknown input and using standard bound-techniques for (12), we get:

$$\begin{aligned} \dot{V} &= -\boldsymbol{x}^\top\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{E}\boldsymbol{x} + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Omega}_1\boldsymbol{u}_2 \\ &\leq -\lambda_{\text{min}}(\boldsymbol{Q})\|\boldsymbol{x}\|^2 + \zeta\lambda_{\text{max}}(\boldsymbol{P})\|\boldsymbol{x}\|^2\|\boldsymbol{E}\| \\ &\quad + \mu_x\lambda_{\text{max}}(\boldsymbol{P})\|\boldsymbol{x}\|^2\|\boldsymbol{\Omega}_1\| \\ \dot{V} &\leq \left(-\lambda_{\text{min}}(\boldsymbol{Q}) + \zeta\lambda_{\text{max}}(\boldsymbol{P})\|\boldsymbol{E}\| + \mu_x\lambda_{\text{max}}(\boldsymbol{P})\|\boldsymbol{\Omega}_1\|\right)\|\boldsymbol{x}\|^2. \end{aligned}$$

Hence, if the networked induced perturbation satisfies the following inequality:

$$\zeta \leq \frac{\lambda_{\text{min}}(\boldsymbol{Q}) - \mu_x\lambda_{\text{max}}(\boldsymbol{P})\|\boldsymbol{\Omega}_1\|}{\lambda_{\text{max}}(\boldsymbol{P})\|\boldsymbol{E}\|},$$

then the origin is a globally exponentially stable equilibrium point. This ends the proof. □

Walsh *et al.* [24] derived a similar bound for the perturbation for the NCS with no unknown inputs. Particularly, the authors is [24] claim that the no-unknown inputs NCS is asymptotically stable if $\|\boldsymbol{e}_x(t)\| \leq \gamma\|\boldsymbol{x}(t)\|$ where γ is a positive constant derived in [24]. This bound is derived for an LTI NCS with no unknown inputs. We can compare our derived bound on the network induced perturbation ζ with the one derived in [24] (γ). This comparison is carried in the simulations section.

VI. More on N$_{\text{et}}$UIO Perturbation Bounds

The objective of this section is to derive a similar bound as the one in [24] for N$_{\text{et}}$UIO. In [24], the authors assumed that all inputs for the plants are known. In this section, we incorporate the effect of the unknown inputs such that we can analyze the stability of the N$_{\text{et}}$UIO, as analyzed in [24].

Lemma 1 (Bellman-Gronwall Lemma [25])**.** *For a given non-negative, piecewise continuous and differentiable functions $\beta(t)$ and $z(t)$, if the function $\boldsymbol{e}_x(t)$ satisfies the following,*

$$\|\boldsymbol{e}_x(t)\| \leq \beta(t) + \int_{t_i}^{t} z(w) \cdot \|\boldsymbol{e}_x(w)\| \, dw \,,$$

then

$$\|\boldsymbol{e}_x(t)\| \leq \beta(t_i) e^{\int_{t_i}^{t} z(s)\, ds} + \int_{t_i}^{t} \dot{\beta}(s) e^{\int_{s}^{t} z(r)\, dr} ds,$$

Proof. The proof can be found in [25]. □

Lemma 2. *For the following representation of the UIO-based NCS,*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\Lambda_1} \boldsymbol{x}(t) + \boldsymbol{E}\boldsymbol{e}_x(t) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t),$$

the following relationship holds,

$$\|\boldsymbol{e}_x(t)\| \leq \epsilon \|\boldsymbol{x}(t)\| + \upsilon,$$

where ϵ and υ are predetermined non-negative constants.

Proof. Recall that the dynamics of the perturbed system can be written as,

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{\Lambda_1}\boldsymbol{x}(t) + \boldsymbol{E}(\boldsymbol{x}(t) - \boldsymbol{x}(t_i)) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t)\\ \dot{\boldsymbol{x}}(t) &= (\boldsymbol{\Lambda_1} + \boldsymbol{E})\boldsymbol{x}(t) - \boldsymbol{E}\boldsymbol{x}(t_i) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t).\end{aligned}$$

Since $\boldsymbol{e}_x(t) = \boldsymbol{x}(t) - \boldsymbol{x}(t_i)$, then following the assumption from [24] that $\dot{\boldsymbol{x}}(t_i) = \boldsymbol{0}$, we have

$$\dot{\boldsymbol{e}}_x(t) = \dot{\boldsymbol{x}}(t) = (\boldsymbol{\Lambda_1} + \boldsymbol{E})\boldsymbol{x}(t) - \boldsymbol{E}\boldsymbol{x}(t_i) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t)$$

$$= (\boldsymbol{\Lambda_1} + \boldsymbol{E})\boldsymbol{x}(t) - (\boldsymbol{\Lambda_1} + \boldsymbol{E})\boldsymbol{x}(t_i) + \boldsymbol{\Lambda_1}\boldsymbol{x}(t_i) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t).$$

Hence, we can write the dynamics of the error as

$$\dot{\boldsymbol{e}}_x(t) = (\boldsymbol{\Lambda_1} + \boldsymbol{E})\boldsymbol{e}_x(t) + \boldsymbol{\Lambda_1}\boldsymbol{x}(t_i) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t).$$

Integrating both sides of the previous equation, we get:

$$\boldsymbol{e}_x(t) = \int_{t_i}^{t} \Big[(\boldsymbol{\Lambda_1} + \boldsymbol{E})\boldsymbol{e}_x(w) + \boldsymbol{\Lambda_1}\boldsymbol{x}(t_i) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(w) \Big] dw.$$

Taking the norm of both sides and applying the submultiplicative property of the matrix norm, and since $t > t_i$ we get

$$\|\boldsymbol{e}_x(t)\| \leq \|\boldsymbol{\Lambda}_1\| \cdot \|\boldsymbol{x}(t_i)\| \cdot (t - t_i)$$

$$+ \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\| \int_{t_i}^{t} \|\boldsymbol{e}_x(w)\| \, dw + \|\boldsymbol{\Omega}_1\| \int_{t_i}^{t} \|\boldsymbol{u}_2(w)\| \, dw \;.$$

Since $\|\boldsymbol{u}_2(w)\| \leq \rho, \ \forall w$ as in [2], we have

$$\|\boldsymbol{e}_x(t)\| \leq \underbrace{\Big(\|\boldsymbol{\Lambda}_1\| \cdot \|\boldsymbol{x}(t_i)\| + \rho \cdot \|\boldsymbol{\Omega}_1\| \Big) \cdot (t - t_i)}_{\beta(t)}$$

$$+ \int_{t_i}^{t} \underbrace{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|}_{z(w)} \cdot \|\boldsymbol{e}_x(w)\| \, dw \,.$$

Applying Lemma 1 for the above inequality, we have

$$\|\boldsymbol{e}_x(t)\| \leq \beta(t_i) e^{\int_{t_i}^{t} z(s)\, ds} + \int_{t_i}^{t} \dot{\beta}(s) e^{\int_{s}^{t} z(r)\, dr} ds,$$

and since $\beta(t_i) = \Big(\|\boldsymbol{\Lambda}_1\| \cdot \|\boldsymbol{x}(t_i)\| + \rho \cdot \|\boldsymbol{\Omega}_1\| \Big) \cdot (t_i - t_i) = 0$, we can write the following:

$$\|\boldsymbol{e}_x(t)\| \leq \int_{t_i}^{t} \dot{\beta}(s) e^{\int_{s}^{t} z(r)\, dr} ds = \tag{13}$$

$$= \Big(\|\boldsymbol{\Lambda}_1\| \cdot \|\boldsymbol{x}(t_i)\| + \rho \cdot \|\boldsymbol{\Omega}_1\| \Big) \int_{t_i}^{t} e^{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\| \cdot (t-s)} ds.$$

We can derive the definite integral in (13) as follows

$$\begin{aligned}&\int_{t_i}^{t} e^{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\| \cdot (t-s)} ds\\ &= -\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} \int_{t_i}^{t} -\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\| e^{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\| \cdot (t-s)} ds\\ &= -\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} e^{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\| \cdot (t-s)} \Big]_{t_i}^{t}\\ &= +\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} (e^{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\| \cdot (t-t_i)} - 1).\end{aligned}$$

Substituting the previous equality in (13), we get:

$$\|\boldsymbol{e}_x(t)\| \leq \tag{14}$$

$$\Big(\|\boldsymbol{\Lambda}_1\| \cdot \|\boldsymbol{x}(t_i)\| + \rho \cdot \|\boldsymbol{\Omega}_1\| \Big) \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} (e^{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\| \cdot (t-t_i)} - 1).$$

Since $\dot{\boldsymbol{x}}(t_i) = \boldsymbol{0}$, then $\boldsymbol{x}(t_i)$ is a constant vector $\forall t \in [t_i, t_{i+1}]$, and hence the vector $\boldsymbol{E}_x = -\boldsymbol{E}\boldsymbol{x}(t_i)$ is a constant vector. Then, we can write the dynamics of the perturbed system as follows, $\dot{\boldsymbol{x}}(t) = (\boldsymbol{\Lambda_1} + \boldsymbol{E})\boldsymbol{x}(t) + \boldsymbol{E}_x + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t)$. Since we want to derive a relationship between $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t_i)$, we can write the following:

$$\boldsymbol{x}(t) = \boldsymbol{x}(t_f) + \int_{t_f}^{t} \Big[(\boldsymbol{\Lambda_1} + \boldsymbol{E})\boldsymbol{x}(w) + \boldsymbol{E}_x + \boldsymbol{\Omega_1}\boldsymbol{u}_2(w) \Big] dw \,.$$

Applying the triangular inequality and submultiplicative matrix norm properties, we can write the following:

$$\|\boldsymbol{x}(t)\| \leq \underbrace{\|\boldsymbol{x}(t_f)\| + \big(\|\boldsymbol{E}_x\| + \rho \cdot \|\boldsymbol{\Omega}_1\| \big) \cdot (t_f - t)}_{\alpha(t)}$$

$$+ \int_{t_f}^{t} \underbrace{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|}_{q(w)} \cdot \|\boldsymbol{x}(w)\| \, dw \,.$$

$$\|\boldsymbol{x}(t_i)\| \leq \frac{\|\boldsymbol{x}(t)\| \cdot e\Big(\|\boldsymbol{\Lambda_1} + \boldsymbol{E}\| \cdot \tau_m\Big) + \rho\|\boldsymbol{\Omega}_1\|\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1}\Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1\Big)}{\Big(1 - \|\boldsymbol{E}\| \cdot \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} \cdot \Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1\Big)\Big)}. \tag{15}$$

$$\|\boldsymbol{e}_x(t)\| \leq \Big(\|\boldsymbol{\Lambda}_1\| \cdot \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1}(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1)\Big)\|\boldsymbol{x}(t_i)\| + \Big(\rho \cdot \|\boldsymbol{\Omega}_1\|\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1}(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1)\Big) \tag{16}$$

Applying Lemma 4 from [24] to the previous inequality, we get:

$$\|\boldsymbol{x}(t)\| \leq \alpha(t_f)e^{\int_t^{t_f} q(s)\,ds} - \int_t^{t_f} \dot{\alpha}(s)e^{\int_t^s z(r)\,dr}ds,$$

which is equivalent to

$$\|\boldsymbol{x}(t)\| \leq \|\boldsymbol{x}(t_f)\|e^{\int_t^{t_f} \|\boldsymbol{\Lambda_1}+\boldsymbol{E}\|\;ds}$$
$$+ \int_t^{t_f} (\|\boldsymbol{E}_x\| + \rho \cdot \|\boldsymbol{\Omega}_1\|) \cdot e\left(\int_t^s \|\boldsymbol{\Lambda_1} + \boldsymbol{E}\|\; dr\right) ds\,.$$

We can write the above inequality as follows,

$$\|\boldsymbol{x}(t)\| \leq \|\boldsymbol{x}(t_f)\| \cdot e\Big(\|\boldsymbol{\Lambda_1} + \boldsymbol{E}\| \cdot (t_f - t)\Big)$$
$$+ (\|\boldsymbol{E}_x\| + \rho \cdot \|\boldsymbol{\Omega}_1\|) \cdot \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} \cdot \Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot(t_f-t)} - 1\Big).$$

Since $\boldsymbol{E}_x = -\boldsymbol{E}\boldsymbol{x}(t_i)$ then $\|\boldsymbol{E}_x\| \leq \|\boldsymbol{E}\| \cdot \|\boldsymbol{x}(t_i)\|$. Letting $t = t_i$ and $t_f = t$, we have

$$\|\boldsymbol{x}(t_i)\| \leq \|\boldsymbol{x}(t)\| \cdot e\Big(\|\boldsymbol{\Lambda_1} + \boldsymbol{E}\| \cdot (t - t_i)\Big)$$
$$+ (\|\boldsymbol{E}\| \cdot \|\boldsymbol{x}(t_i)\| + \rho \cdot \|\boldsymbol{\Omega}_1\|) \cdot \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1}$$
$$\cdot\Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot(t-t_i)} - 1\Big).$$

Since MATI is defined as $\tau_m = t - t_i$, we can write the previous inequality as:

$$\Big(1 - \|\boldsymbol{E}\| \cdot \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} \cdot \Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1\Big)\Big)\|\boldsymbol{x}(t_i)\| \leq$$
$$\|\boldsymbol{x}(t)\| \cdot e\Big(\|\boldsymbol{\Lambda_1} + \boldsymbol{E}\| \cdot \tau_m\Big)$$
$$+\rho\|\boldsymbol{\Omega}_1\|\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1}\Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1\Big).$$

or as in (15), as shown on top of the page. We can also rewrite (14) as in (16). Substituting (15) in (16), we get the following relationship:

$$\|\boldsymbol{e}_x(t)\| \leq \epsilon\,\|\boldsymbol{x}(t)\| + \upsilon, \tag{17}$$

where ϵ and υ are predetermined constants, as follows:

$$\epsilon = \frac{\Big(\|\boldsymbol{\Lambda}_1\| \cdot \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1}(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1)\Big) \cdot e^{\|\boldsymbol{\Lambda_1}+\boldsymbol{E}\|\cdot\tau_m}}{\Big(1 - \|\boldsymbol{E}\| \cdot \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} \cdot \Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1\Big)\Big)}$$

$$\upsilon = \frac{\Big(\|\boldsymbol{\Lambda}_1\| \cdot (e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1)\Big) \cdot \rho\|\boldsymbol{\Omega}_1\|\Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1\Big)}{\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^2\Big(1 - \|\boldsymbol{E}\| \cdot \|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1} \cdot \Big(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1\Big)\Big)}$$
$$+\rho \cdot \|\boldsymbol{\Omega}_1\|\|\boldsymbol{\Lambda}_1 + \boldsymbol{E}\|^{-1}(e^{\|\boldsymbol{\Lambda}_1+\boldsymbol{E}\|\cdot\tau_m} - 1).$$

Notice that by setting $\rho = 0$ (i.e., ignoring the effect of the unknown input), the relationship between $\|\boldsymbol{e}_x(t)\|$ and $\|\boldsymbol{x}(t)\|$ reduces to the same relationship as in [24]. ☐

This derived bound gives us a relationship between the norm of the networked-induced error and the plant-state, paving the way to assess the stability of the N_{et}UIO through applying traditional Lyapunov stability theory.

Theorem 2. *For the perturbed N_{et}UIO as represented in (10) and for a Hurwitz $\boldsymbol{\Lambda}_1$, we have $\boldsymbol{P} = \boldsymbol{P}^\top \succ \boldsymbol{O}$, is the solution to the Lyapunov matrix equation*

$$\boldsymbol{\Lambda}_1^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{\Lambda}_1 = -2\boldsymbol{Q},$$

for a given $\boldsymbol{Q} = \boldsymbol{Q}^\top \succ \boldsymbol{O}$. If $\|\boldsymbol{u}_2\| < \mu_x\|\boldsymbol{x}\|$, where $\mu_x > 0$, and if

$$1 - ||\boldsymbol{E}||\;||\boldsymbol{\Lambda}_1 + \boldsymbol{E}||^{-1}(e^{||\boldsymbol{\Lambda}_1+\boldsymbol{E}||\tau_m} - 1) > 0$$

then if the norm of the network induced perturbation $\|\boldsymbol{e}_x\|$ satisfies $\|\boldsymbol{e}_x\| < \epsilon\|\boldsymbol{x}\| + \upsilon$ where ϵ and υ are non-negative constants derived in Lemma 2, then the origin is a globally exponentially stable equilibrium point of the N_{et}UIO if

$$\mu_x \leq \frac{\lambda_{min}(\boldsymbol{Q}) - \epsilon\|\boldsymbol{P}\boldsymbol{E}\|}{\|\boldsymbol{P}\boldsymbol{\Omega}_1\|}.$$

Proof. We now use the derived bound to establish bounds on the stability of the perturbed representation of the closed loop system. Since $\boldsymbol{\Lambda}_1$ is asymptotically stable, the solution $\boldsymbol{P}$ to the Lyapunov matrix equation

$$\boldsymbol{\Lambda}_1^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{\Lambda}_1 = -2\boldsymbol{Q},$$

is symmetric positive definite (i.e., $\boldsymbol{P} = \boldsymbol{P}^\top \succ \boldsymbol{O}$.) for a given $\boldsymbol{Q} = \boldsymbol{Q}^\top \succ \boldsymbol{O}$.

Let $V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{x}$ be a Lyapunov candidate function, used to verify stability and establish bounds on the network-induced perturbation. Particularly, we need to find a bound on the perturbation $\|\boldsymbol{e}_x\|$ such that $\dot{V}(\boldsymbol{x}) < 0$. Taking the derivative of the Lyapunov candidate function and applying the bounds on the unknown input and the networked induced error (Lemma 2), we get,

$$\dot{V}(\boldsymbol{x}) = \boldsymbol{x}^\top\boldsymbol{P}\dot{\boldsymbol{x}} \tag{18}$$

Substituting (10) into (18), we get:

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \boldsymbol{x}^\top \boldsymbol{P}(\boldsymbol{\Lambda}_1\boldsymbol{x} + \boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{\Omega}_1\boldsymbol{u}_2) \\ &= \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Lambda}_1\boldsymbol{x} + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Omega}_1\boldsymbol{u}_2 \\ &= \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Lambda}_1\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Lambda}_1\boldsymbol{x} \\ &\quad + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Omega}_1\boldsymbol{u}_2 \\ &= \frac{1}{2}\boldsymbol{x}^\top\left(\boldsymbol{P}\boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_1^\top\boldsymbol{P}\right)\boldsymbol{x} + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Omega}_1\boldsymbol{u}_2 \\ &= -\boldsymbol{x}^\top\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_x + \boldsymbol{x}^\top\boldsymbol{P}\boldsymbol{\Omega}_1\boldsymbol{u}_2 \\ &\leq \underbrace{(-\lambda_{\min}(\boldsymbol{Q}) + \epsilon\|\boldsymbol{P}\boldsymbol{E}\| + \mu_x\|\boldsymbol{P}\boldsymbol{\Omega}_1\|)}_{-\pi_1}\|\boldsymbol{x}\|^2 \\ &\quad + \underbrace{\upsilon\|\boldsymbol{P}\boldsymbol{E}\|}_{\pi_2}\|\boldsymbol{x}\|\end{aligned}$$

For most systems, the assumption $\pi_1\|\boldsymbol{x}\| - \pi_2 > 0$ holds as $\pi_1 >> \pi_2$. Hence, the origin is a globally exponentially stable equilibrium point of the N_{et}UIO if

$$\mu_x \leq \frac{\lambda_{\min}(\boldsymbol{Q}) - \epsilon\|\boldsymbol{P}\boldsymbol{E}\|}{\|\boldsymbol{P}\boldsymbol{\Omega}_1\|}.$$

This ends the proof. □

VII. Numerical Results

In this section, we test the derived bounds from Sections V and VI on the perturbation for the N_{et}UIO. The simulated numerical example is a stable LTI SISO system with one known and one unknown inputs and one output (i.e., $n = 3, m_1 = 1, m_2 = 1, p = 1$). The system is modeled by:

$$\boldsymbol{A} = \boldsymbol{A}_p = \begin{bmatrix}-5 & 3 & 0\\ 4 & -10 & 4\\ 0 & 0 & -4\end{bmatrix}, \boldsymbol{B}_1 = \boldsymbol{B}_p^{(1)} = \begin{bmatrix}1\\0\\2\end{bmatrix},$$

$$\boldsymbol{B}_2 = \boldsymbol{B}_p^{(2)} = \begin{bmatrix}2\\1\\2\end{bmatrix}, \;\&\; \boldsymbol{C} = \boldsymbol{C}_p = \begin{bmatrix}2 & 4 & -1\end{bmatrix}.$$

We first find the non-networked UIO parameters ($\boldsymbol{A}_c, \boldsymbol{B}_c^{(1)}, \boldsymbol{B}_c^{(2)}, \boldsymbol{C}_c$ and $\boldsymbol{D}_c$):

$$\boldsymbol{A}_c = \begin{bmatrix}-7 & 14.3 & -6.6\\ 3 & -4.3 & 0.6\\ -2 & 11.3 & -10.6\end{bmatrix}, \boldsymbol{B}_c^{(1)} = \begin{bmatrix}-2.1\\0.5\\-2.3\end{bmatrix}, \boldsymbol{B}_c^{(2)} = \begin{bmatrix}1\\0\\2\end{bmatrix},$$

$$\boldsymbol{C}_c = \begin{bmatrix}-36 & -16.5 & 5\end{bmatrix}, \boldsymbol{D}_c = -13.$$

The initial plant and UIO's states are chosen differently from each others. The unknown input includes disturbances and actuator faults and is given by:

$$u_2(t) = \sin(t) + \cos(t) + \max\{0, t - 0.5\}.$$

Figure 1 shows the UIO state-estimation error for the non-networked system. Note that the state estimation error converges rapidly to zero.

We now simulate the networked system. Using the parameters for the plant and the non-networked UIO, we compute $\boldsymbol{\Lambda_1}, \boldsymbol{\Lambda_2}, \boldsymbol{\Omega}_1$ and $\boldsymbol{E}$. In Section V, we show that if the norm

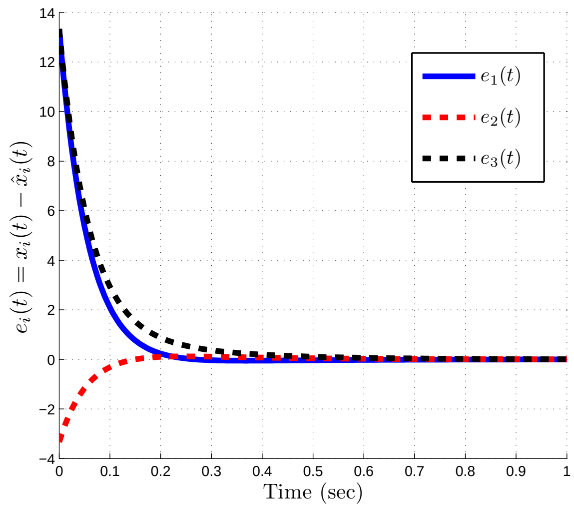

Fig. 1. UIO State Estimation Error for Non-Networked System with unknown input $u_2(t) = \sin(t) + \cos(t) + \max\{0, t - 0.5\}$. State estimation error converges rapidly to zero.

of the network induced perturbation satisfies $\|\boldsymbol{e}_x\| < \zeta\|\boldsymbol{x}\|$ where

$$\zeta \leq \frac{\lambda_{\min}(\boldsymbol{Q}) - \mu_x\|\boldsymbol{\Omega_1}\|\lambda_{\max}(\boldsymbol{P})}{\lambda_{\max}(\boldsymbol{P})\|\boldsymbol{E}\|},$$

the origin is a globally exponentially stable equilibrium point of the perturbed N_{et}UIO of the following form,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\Lambda_1}\boldsymbol{x}(t) + \boldsymbol{E}\boldsymbol{e}_x(t) + \boldsymbol{\Omega_1}\boldsymbol{u}_2(t).$$

We apply the result from Section V to the SISO system. After finding a matrix $\boldsymbol{P} = \boldsymbol{P}^\top \succ \boldsymbol{O}$, that is a solution to the Lyapunov matrix equation

$$\boldsymbol{\Lambda}_1^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{\Lambda}_1 = -2\boldsymbol{Q},$$

for a given $\boldsymbol{Q} = \boldsymbol{Q}^\top \succ \boldsymbol{O}$ and for $\mu_x = 0.1$, the theoretical bound that would render the system unstable is $\|\boldsymbol{e}_x\| < 1.2097 \cdot \|\boldsymbol{x}\|$. for a τ_m that satisfies the aforementioned inequality. Numerical results for the bound derived in [24] show that the SISO N_{et}UIO-based NCS is unstable for $\|\boldsymbol{e}_x\| > 1.42 \cdot 10^{-4}\|\boldsymbol{x}\|$, which is too conservative compared to the bound derived (and computed: $\|\boldsymbol{e}_x\| < 1.2097 \cdot \|\boldsymbol{x}\|$) in Section V.

Choosing an unknown input as the one before and a random network error disturbance that satisfies the above derived bounds, we simulate the networked system with the same initial conditions as for the non-networked system. Figure 2 shows the state estimates for the networked system. The state estimates converges to the actual plant states. This shows the applicability of the derived bounds on the unknown input and network disturbance.

VIII. Conclusions and Future Work

In many control systems, observers' and state estimators' inputs are usually sent through a communication network, which is a key component in modern Networked Control

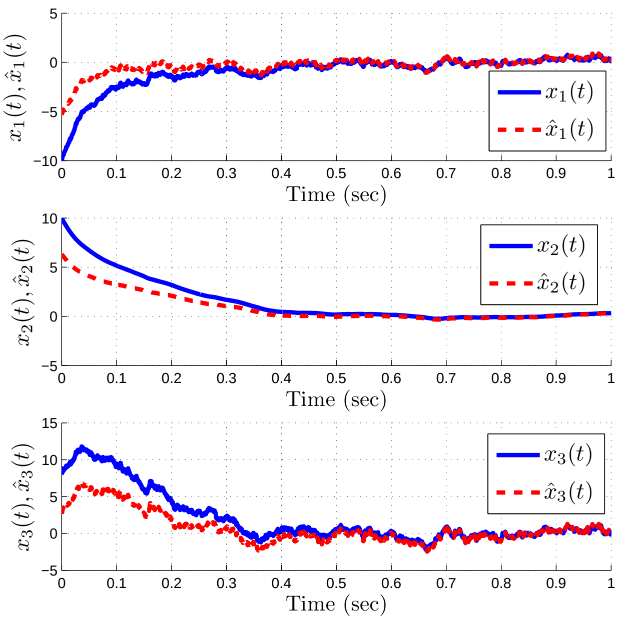

Fig. 2. UIO State Estimates for the Networked System with Random Disturbances

Systems. The objective of the research presented in this paper is to analyze the effects of plant unknown inputs and network disturbances for Unknown Input Observer architecture. First, we review an UIO design for non-networked system from the UIO literature. Second, we derive the dynamics of the UIO-based NCS given that the network effect is viewed as perturbation the exchanged signals through the communication network. Third, we derive bounds on the network perturbation and unknown inputs for the N_{et}UIO. Finally, numerical simulations are given to highlight the applicability of the formulated bounds. The results illustrate that the derived perturbation bounds for N_{et}UIO are accurate.

The derivation of unknown inputs and network disturbance perturbation bounds would improve the design of controllers and observers such that the overall system performs desirably. For example, the state-feedback gain and UIO gain matrices can be designed to reduce the disturbance effects of unknown inputs and network perturbation. In our future work, we plan to apply the proposed framework to different Cyber-Physical Systems and further investigate the applicability of the proposed bounds to real-life applications.

Acknowledgments

The authors gratefully acknowledge the financial support from the National Science Foundation through NSF CMMI Grants 1201114 and 1265622.

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