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An Integral Function Approach to the Exponential Stability of Linear Time-Varying Systems

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Abstract: This paper studies the exponential stability of linear time-varying (LTV) systems using the recent proposed integral function. By showing the properties of the integral function and applying the Bellman-Gronwall Lemma, a sufficient and necessary condition for the exponential stability of LTV systems is derived. Furthermore, the exponential decay rate of the system trajectories can be obtained by computing the radii of convergence of integral function. The algorithm for computing the integral function is also developed and two classical examples are given to illustrate the proposed approach.

Keywords: Bellman-Gronwall Lemma, exponential stability, integral function, linear time-varying systems.

1. INTRODUCTION

The linear continuous time-varying (LTV) systems has been receiving increasing attention by system and control community, since they appear frequently in practical engineering areas such as aerospace control systems [1,2]. While important, LTV systems are very hard to investigate despite of the fundamental stability problem. It is well known that, even when the eigenvalues of the system have strictly negative real parts for all instants of time, the time-varying system may be unstable.

However, numerous important progresses, including but not limited to [3-10] have been achieved through the effort of researchers. They more or less all rely on the use of a linear time-invariant plant as an approximation of the LTV system and ensuring that the influence of the approximation is not excessive. The main advantage of this frozen time method is the possibility of exploiting the great deal of tools which have been developed for linear time-invariant (LTI) systems.

This paper tries to present a novel approach to investigate the exponential stability of LTV systems. In the previous work [11], an integral function approach was proposed to analyze the exponential stability of a class of piecewise-linear systems, and a computational sufficient and necessary criterion was provided in terms of the integral function.

In this paper, an improved integral function is introduced, which has some nice properties including homogeneity, sub-additivity, convexity, common-bound and vertex-bound. Based on the properties and Bellman-Gronwall lemma, a sufficient and necessary condition for the exponential stability of LTV systems is derived, and the exponential decay rate of the LTV systems is characterized by the radius of convergence of integral function without conservatism. As our best knowledge of LTV systems, it is the first time that such a rate has been characterized exactly.

2. PROBLEM FORMULATION

We consider a class of continuous-time PPLS represented by

$$\dot{x}(t) = A(t)x(t), \quad t \ge t_0, \tag{1}$$

where $x(t) \in \mathrm{R}^n$ and $A(t) \in \mathrm{R}^{n\times n}$ are. It is assumed that $A(t)$ is continuous in t, and bounded for all $t \ge t_0$.

Definition 1: Let $x(t,t_0;z)$ denotes the solution of LTV system with initial time t_0 and initial state z. The system (1) is called exponentially stable if there exist $r \in (0,1)$ and $\kappa_r > 0$ such that $x(t,t_0;z) \le \kappa_r r^{t-t_0} \|z\|$ for all $t \ge t_0$.

Definition 2: Define the exponential decay rate of LTV system (1) as

$$r^* = \inf\{r \mid x(t,t_0;z) \le \kappa_r r^{t-t_0} \|z\|, \ z \in \mathrm{R}^n, \ t \ge t_0\} \tag{2}$$

to characterize the convergence rate of the "most unstable" trajectories of LTV system (1).

The objectives are: (i) present a computable sufficient and necessary criterion of exponential stability for LTV systems; (ii) compute the exponential decay rate without conservatism.

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Lemma 1: The solution of LTV system (1) has the following properties:

$$x(t,t_0;\alpha z)=\alpha x(t,t_0;z), \quad \alpha\in\mathrm{R};$$
$$x(t,t_0;z_1+z_2)=x(t,t_0;z_1)+x(t,t_0;z_2), \quad z_1,z_2\in\mathrm{R}^n.$$

3. MAIN RESULTS

In this section, we present the integral function approach to the exponential stability analysis of LTV systems. The integral function is defined, analyzed and used to determine the exponential stability of system (1). Furthermore, the radii of convergence of integral function is defined and used to compute the exponential decay rate of LTV systems. Additionally, the algorithm for computing the integral function is developed such that the whole approach is computationally effective.

Definition 3: We define *integral function* $I(\cdot,z)$ of the LTV system as:

$$I(\lambda,z)=\sup_{t_0\ge 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\left\|x(t,t_0;z)\right\|^2 dt. \tag{3}$$

For each fixed $\lambda\ge 0$, $I(\lambda,z)$ is a function of z only:

$$I_\lambda(z):=I(\lambda,z). \tag{4}$$

Obviously, $I_\lambda(0)=0$, and $I_\lambda(z_1)\ge I_\lambda(z_2)$, $z\ne 0$ if and only if $\lambda_1>\lambda_2\ge 0$.

3.1. Properties of integral function

Based on Definition 3 and Lemma 1, we obtain the following properties of the integral function.

Proposition 1: $I_\lambda(z)$ has the following properties:

1. (Homogeneity) $I_\lambda(z)$ is homogeneous of degree two in z, i.e., $I_\lambda(\alpha z)=\alpha^2 I_\lambda(z)$, $\alpha>0$.

2. (Sub-Additivity) For all $z_1,z_2\in\mathrm{R}^n$,

$$\sqrt{I_\lambda(z_1+z_2)}\le\sqrt{I_\lambda(z_1)}+\sqrt{I_\lambda(z_2)}. \tag{5}$$

3. (Convexity) For each $\lambda\ge 0$, $\sqrt{I_\lambda(z_1)}$ is a convex function, i.e., for all $z_1,z_2\in\mathrm{R}^n$ and $\alpha_1,\alpha_2\ge 0$, $\alpha_1+\alpha_2=1$,

$$\sqrt{I_\lambda(\alpha_1 z_1+\alpha_2 z_2)}\le\alpha_1\sqrt{I_\lambda(z_1)}+\alpha_2\sqrt{I_\lambda(z_2)}. \tag{6}$$

4. (Common-Bound) For each $\lambda\ge 0$, $I_\lambda(z)<\infty$ for all $z\in\mathrm{R}^n$ implies that $I_\lambda(z)<c\|z\|^2$ for some c.

5. (Vertex-Bound) $I_\lambda(z_i)<\infty$, $i\in\{1,\cdots,n\}$ implies that, $I_\lambda(z)<\infty$ for all $z\in\mathrm{R}^n$, where $\{z_i\}_{i=1}^n$ is a standard basis of R^n.

Proof: 1. The homogeneity property is a direct sequence of homogeneity property of system solution (see Lemma 1).

2. It can be implied by Lemma~1 and Cauchy-Schwartz inequality in the integral form [12],

$$\begin{aligned}
I_\lambda(z_1+z_2)&=\sup_{t_0\ge 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\left\|x(t,t_0;z_1+z_2)\right\|^2 dt\\
&=\sup_{t_0\ge 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\left\|x(t,t_0;z_1)+x(t,t_0;z_2)\right\|^2 dt\\
&=\sup_{t_0\ge 0}\int_{t_0}^{\infty}\lambda^{t-t_0}[\left\|x(t,t_0;z_1)\right\|^2+\left\|x(t,t_0;z_2)\right\|^2\\
&\quad+2\left\|x(t,t_0;z_1)\right\|\cdot\left\|x(t,t_0;z_2)\right\|]dt\\
&\le\sup_{t_0\ge 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\left\|x(t,t_0;z_1)\right\|^2 dt\\
&\quad+\sup_{t_0\ge 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\left\|x(t,t_0;z_2)\right\|^2 dt\\
&\quad+2\sqrt{\sup_{t_0\ge 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\left\|x(t,t_0;z_1)\right\|^2 dt}\\
&\quad\cdot\sqrt{\sup_{t_0\ge 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\left\|x(t,t_0;z_2)\right\|^2 dt}\\
&=I_\lambda(z_1)+I_\lambda(z_2)+2\sqrt{I_\lambda(z_1)}\cdot\sqrt{I_\lambda(z_2)}\\
&=\left(\sqrt{I_\lambda(z_1)}+\sqrt{I_\lambda(z_2)}\right)^2.
\end{aligned}$$

This implies the result (5).

3. With the help of Sub-Additivity and Homogeneity for $\alpha_1,\alpha_2\ge 0$ and $\alpha_1+\alpha_2=1$, we have

$$\begin{aligned}
\sqrt{I_\lambda(\alpha_1 z_1+\alpha_2 z_2)}&\le\sqrt{I_\lambda(\alpha_1 z_1)}+\sqrt{I_\lambda(\alpha_2 z_2)}\\
&=\alpha_1\sqrt{I_\lambda(z_1)}+\alpha_2\sqrt{I_\lambda(z_2)}.
\end{aligned}\tag{7}$$

This proves the Convexity of $I_\lambda(z)$.

4. Assume there exist $\lambda\ge 0$ such that $I_\lambda(z)<\infty$. Let $\{z_i\}_{i=1}^n$ denote a standard basis of R^n, then for any $z\in\mathrm{S}^{n-1}$ (unit-ball of R^n), there exist $\alpha_j\ge 0$, $\sum_{j=1}^n\alpha_j^{\,2}=1$, such that

$$z=\sum_{j=1}^n\alpha_j z_j\,. \tag{8}$$

Apply the Sub-Additivity of $I_\lambda(z)$ and the Cauchy-Schwartz inequality in the summation form [12] to get that, for all $z\in\mathrm{S}^{n-1}$

$$\begin{aligned}
I_\lambda(z)&=\sqrt{I_\lambda\left(\sum_{j=1}^n\alpha_j z_j\right)}\le\left[\sum_{j=1}^n\alpha_j\sqrt{I_\lambda\left(z_j\right)}\right]^2\\
&\le\sum_{j=1}^n\alpha_j^{\,2}.\sum_{j=1}^n I_\lambda\left(z_j\right)\le c,
\end{aligned}\tag{9}$$

where

$$c := n \cdot \max_{j\in\{1,\cdots,n\}} I_\lambda(z_j). \tag{10}$$

By homogeneity, we have $I_\lambda(z) < c\|z\|^2$, $z \in \mathrm{R}^n$.

5. The property 5 follows directly from (9-10).

Remark 1: Mathematically speaking, given the LTV system (1), property 4 actually shows that the pointwise boundness of the integral function implies a common upper bound of integral function. This property is key important to the exponential stability analysis of the LTV system. Moreover, it can be learned from the proof of property 4 that the proposed upper bound is fully determined by the integral function values on the standard basis, i.e., $I_\lambda(z_i)_{i=1}^n$, which yields that the integral function approach is computationally effective.

3.2. Exponential stability criterion of LTV systems

In this subsection, a sufficient and necessary condition of exponential stability is presented for LTV systems (1) via the proposed integral function.

Theorem 1: Consider the LTV system (1), the following statements are equivalent.

(1) The LTV system (1) is exponentially stable.

(2) There exist $\lambda > 1$ and a standard basis $\{z_i\}_{i=1}^n$ such that the integral function on the standard basis is finite, i.e., $I_\lambda(z_i) < \infty,\ i \in \{1,\cdots,n\}$.

(3) There exists $\lambda > 1$ such that the integral function $I_\lambda(z)$ is finite for all $z \in \mathrm{R}^n$.

Proof: We begin by $(1) \Rightarrow (2)$. Suppose the LTV system (1) is exponentially stable, i.e., there exists constants $\kappa \geq 1$ and $r \in (0,1)$ such that

$$x(t,t_0;z) \leq \kappa_r r^{t-t_0}\|z\|, \quad t\in[0,\infty). \tag{11}$$

By this condition we have, for all $z \in \mathrm{R}^n$

$$\begin{aligned} I_\lambda(z) &= \sup_{t_0\geq 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\|x(t,t_0;z)\|^2\,dt \\ &\leq \sup_{t_0\geq 0}\int_{t_0}^{\infty}\kappa^2(\lambda r^2)^{t-t_0}\|z\|^2\,dt \\ &= \int_0^{\infty}\kappa^2(\lambda r^2)^t\|z\|^2\,dt \\ &= \frac{-\kappa^2}{2\ln(\lambda r^2)}\|z\|^2 < \infty. \end{aligned} \tag{12}$$

Note that $(2) \Rightarrow (3)$ is a direct conclusion of property 5 in Proposition 1. Thus, the rest work is to show $(3) \Rightarrow (1)$.

Assume $\lambda > 1$ such that $I_\lambda(z)$ is finite. By the property 4 in Proposition 1, we learn that, there exists constant $c > 0$ such that

$$I_\lambda(z) = \sup_{t_0\geq 0}\int_{t_0}^{\infty}\lambda^{t-t_0}\|x(t,t_0;z)\|^2\,dt \leq c,\ \forall z\in \mathrm{S}^{n-1}. \tag{13}$$

This implies that, for any given $\varepsilon > 0$, there exists $T_0 > 0$ such that for all $t_0 \geq 0$ and $z \in \mathrm{S}^{n-1}$

$$\int_{t_0+T}^{t_0+T+1}\lambda^{t-t_0}\|x(t,t_0;z)\|^2\,dt < \varepsilon, \quad T > T_0. \tag{14}$$

Applying First Mean Value Theorem [12] to (14), we have that, there exists $t_* \in [T, T+1]$ such that

$$\int_{t_0+T}^{t_0+T+1}\lambda^{t-t_0}\|x(t,t_0;z)\|^2\,dt = \lambda^{t_*}\|x(t_0+t_*,t_0;z)\| < \varepsilon. \tag{15}$$

On the other hand, for all $t \geq 0$,

$$\begin{aligned} &\|x(t_0+t,t_0;z)\|^2 \\ &= \int_{t_*}^{t}\frac{d}{d\tau}\left(\|x(t+\tau,t_0;z)\|^2\right)d\tau + \|x(t_0+t_*,t_0;z)\|^2 \\ &= 2\int_{t_*}^{t}x(t+\tau,t_0;z)^T A(t_0+\tau)x(t+\tau,t_0;z)d\tau \\ &\quad + \|x(t_0+t_*,t_0;z)\|^2 \\ &\leq 2\varsigma\int_{t_*}^{t}\|x(t+\tau,t_0;z)\|^2 d\tau + \|x(t_0+t_*,t_0;z)\|^2, \end{aligned} \tag{16}$$

where ς is the upper bound of $\|A(t)\|$.

By Bellman-Gronwall Lemma [13], one thus has

$$\|x(t_0+t,t_0;z)\|^2 \leq \mathrm{e}^{2\varsigma|t-t_*|}\|x(t_0+t_*,t_0;z)\|^2. \tag{17}$$

In particular, for $t \in [T, T+1]$,

$$\|x(t_0+t,t_0;z)\|^2 \leq \mathrm{e}^{2\varsigma}\|x(t_0+t_*,t_0;z)\|^2. \tag{18}$$

With the help of (15), we can further obtain that

$$\begin{aligned} &\|x(t_0+t,t_0;z)\|^2 \leq \mathrm{e}^{2\varsigma}\varepsilon\lambda^{-t_*} = \varepsilon\mathrm{e}^{2\varsigma}\lambda^{t-t_*}\lambda^{-t} \\ &\leq \varepsilon\mathrm{e}^{2\varsigma}\lambda^{-t+1}, \quad t\in[T,T+1], \quad T>T_0. \end{aligned} \tag{19}$$

By noting that the ε, λ and ς are all independent with T to get

$$\|x(t_0+t,t_0;z)\|^2 \leq \varepsilon\lambda\mathrm{e}^{2\varsigma}\lambda^{-t}, \quad \forall t > T_0. \tag{20}$$

Let $M = \sup_{t_0\in[0,\infty)}\sup_{t\in[0,T_0]}\|x(t_0+t,t_0;z)\|^2$, then we have $M < \infty$ in view of (13) and (17), hence for all $z \in \mathrm{S}^{n-1}$,

$$\|x(t_0+t,t_0;z)\|^2 \leq M(1+\varepsilon)\lambda^{1+T_0}\mathrm{e}^{2\varsigma}\lambda^{-t}, \quad \forall t\geq 0 \tag{21}$$

i.e.,

$$\|x(t,t_0;z)\|^2 \leq M(1+\varepsilon)\lambda^{1+T_0}\mathrm{e}^{2\varsigma}\lambda^{-(t-t_0)}, \quad \forall t\geq t_0. \tag{22}$$

Furthermore, by denoting $\kappa = M(1+\varepsilon)\lambda^{1+T_0}\mathrm{e}^{2\varsigma}$ and $r = \lambda^{-1/2}$, the result (22) can be reformulated as

$$\|x(t,t_0;z)\|^2 \leq \kappa\lambda^{-(t-t_0)}, \quad \forall t\geq t_0, z\in \mathrm{S}^{n-1}. \tag{23}$$

At last, applying the Homogeneity of system solution and noting that $r \in (0,1)$, we obtain the result that the LTV system is exponentially stable.

Remark 2: From Theorem 1, to judge the exponential stability of LTV system (1), we just need to check if there exists $\lambda > 1$ such that the integral function $I_\lambda(z)$ on a standard basis $\{z_i\}_{i=1}^{n}$ is finite, which can be effectively solved using the algorithm to be developed later.

3.3. Characterization of exponential decay rate

In the following subsection, we define the radius of convergence of the integral function. It can be shown this quantity characterizes the exponential decay rate of the LTV system.

Definition 4: The radius of convergence of integral function denoted by λ^*, is defined as

$$\lambda^* = \sup\left\{\lambda \geq 0 \mid I_\lambda(z) < \infty, \quad \forall z \in \mathrm{R}^n\right\}. \tag{24}$$

The following theorem shows the relationship between the radius of convergence λ^* and the exponential decay rate r^*.

Theorem 2: Given the LTV system (1) with a radius of convergence λ^*, then for any $r > (\lambda^*)^{-1/2}$, there exists a constant κ_r such that $\|x(t,t_0;z)\| \leq \kappa_r r^{t-t_0}\|z\|$, $t \geq t_0$. Furthermore, $(\lambda^*)^{-1/2}$ is also the smallest value for the previous statement to hold. In other words, the exponential decay rate $r^* = (\lambda^*)^{-1/2}$.

The proof is similar with $(1) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ in the proof of Theorem 1 by considering the scaled LTV system with system matrix $A(t)/r$, therefore we skip the proof here.

3.4. Computation of integral function

All the analysis methods proposed in the previous subsections require to compute the integral functions of LTV systems on a standard basis $\{z_i\}_{i=1}^{n}$. Therefore, we develop an algorithm for computing the truncation of integral functions as the approximations of $I_\lambda(z)$.

Definition 5: For each $T > 0$, define

$$I_\lambda^T(z) = \sup_{0 \leq t_0 \leq T} \int_{t_0}^{T} \lambda^{t-t_0} \|x(t,t_0;z)\|^2 \, dt. \tag{25}$$

The following proposition shows the relationship between $I_\lambda^T(z)$ and $I_\lambda(z)$.

Proposition 2: Consider the LTV system (1), the following two statements hold.

1. For $\lambda > 0$, the integral function $I_\lambda(z)$ is infinite for some $z \in \mathrm{R}^n$, then $I_\lambda^T(z)$ will converge to infinite with the increase of T.

2. For $\lambda > 0$, the integral function $I_\lambda(z)$ is finite for all $z \in \mathrm{R}^n$, then $\lim_{T\to\infty} I_\lambda^T(z) = I_\lambda(z)$.

Proof: 1. The apagoge is employed. Assume that there exists $z \in \mathrm{R}^n$, $I_\lambda(z)$ is infinite, but $I_\lambda^T(z)$ is bounded in $T \in (0,\infty)$, i.e., there exists $M > 0$ such that for all $z \in \mathrm{S}^{n-1}$, $I_\lambda^T(z) \leq M, \forall T \geq 0$. This implies that, for all $t_0 \in [0,T]$ and $z \in \mathrm{S}^{n-1}$

$$\int_{t_0+T}^{t_0+T+1} \lambda^{t-t_0} \|x(t,t_0;z)\|^2 \, dt < M, \quad \forall T > T_0. \tag{26}$$

By the similar deduction with the proof for $(3) \Rightarrow (1)$ in Theorem 1, we learn that, there exists constant $c > 0$ such that

$$\|x(t,t_0;z)\| \leq C\lambda^{-\frac{1}{2}(t-t_0)}, \quad t \geq t_0. \tag{27}$$

From Theorem 2, the above condition implies that $\lambda^{-1/2} > (\lambda^*)^{-1/2}$, which further yields $\lambda < \lambda^*$. However, this is in contradiction with the assumption that $I_\lambda(z)$ is infinite.

2. For $\lambda < \lambda^*$, let $\lambda_1 = (\lambda + \lambda^*)/2$, then we have, $I_{\lambda_1}(z) < \infty$, $z \in \mathrm{R}^n$. It can be learned from Theorem 2, there exists constant $\kappa > 0$ such that

$$\|x(t,t_0;z)\| \leq \kappa\lambda_1^{-\frac{1}{2}(t-t_0)}\|z\|^2, \quad t \geq t_0. \tag{28}$$

With the help of this inequality, we have

$$\begin{aligned} \left\|I_\lambda^T(z) - I_\lambda(z)\right\| &\leq \int_T^\infty \lambda^t \|x(t,t_0;z)\|^2 \, dt \\ &\leq \kappa^2 \int_T^\infty \lambda^{(t-t_0)}\lambda_1^{-(t-t_0)} \|z\|^2 \, dt \\ &= \frac{\kappa^2}{\ln(\lambda_1/\lambda)}\left(\frac{\lambda}{\lambda_1}\right)^T \|z\|^2. \end{aligned} \tag{29}$$

Note that $0 < \lambda < \lambda_1$, then letting T converges to infinity, we have, for all $z \in \mathrm{R}^n$,

$$\lim_{T\to\infty} I_\lambda^T(z) = I_\lambda(z). \tag{30}$$

From the above proposition, an approximation of the integral function $I_\lambda(z)$ is provided by $I_\lambda^T(z)$ for T large enough, which can be obtained by the following algorithm.

Algorithm 1: Computation of $I_\lambda(z)$

1. **Initialize** $T = 1, I_\lambda^{T/2} = 0$ and $\varepsilon = 0.01$;
2. **Repeat** $T \leftarrow 2T$;
3. **Set** $\zeta = 10T$;
4. **Repeat** $\zeta \leftarrow 2\zeta$;
5. **Set** $\tau = \zeta/T, M = T/\zeta$;
6. **for** each $k \in \{0,\cdots,M\}$, **do**
7. $t_0 = k\tau$;
8. $\left[\{x(t_i)\}_{i=1}^{2N+1}\right] = \text{ode45}(LTV, t_0, z, [t_0, t_0+T])$;
9. $I_\lambda^T(z) = \frac{T}{6N}\sum_{i=1}^{N}\left[x(t_{2i-1}) + 4x(t_{2i}) + x(t_{2i+1})\right]$
10. **end for**
11. **Until** $\left\|I_\lambda^{T,\zeta}(z) - I_\lambda^{T,\zeta/2}(z)\right\| < \varepsilon$;
12. **Set** $I_\lambda^T(z) = I_\lambda^{T,\zeta}(z)$;
13. **Until** $\left\|I_\lambda^T(z) - I_\lambda^{T/2}(z)\right\| < \varepsilon$;
14. **Return** $I_\lambda^T(z)$.

By increasing simulation time T and decreasing the step-length τ, we can obtain the underestimates of $I_\lambda(z)$ with any precision as permitted by the numerical computation errors. Therefore, the exponential stability of LTV system can be judged by checking the values of $I_\lambda(z)$ on the standard basis $\{z_i\}_{i=1}^n$. Furthermore, by computing $I_\lambda = \max_{i\in\{1,\cdots,n\}} I_\lambda(z)$ for an increasing sequence of $\lambda > 0$, an underestimate of λ^* can also be obtained.

4. ILLUSTRATIVE EXAMPLES

To illustrate the established results, two classical examples from the literatures are considered.

Example 1: Consider a mass-spring system where both damping and elastic constant coefficients are time-varying, as presented for instance in [8,10]. Let α be a positive constant parameter. The system matrix is given by

$$A(t) = \begin{bmatrix} 0 & 1 \\ -(2-\alpha\sin(t)) & -(2-\alpha\cos(t)) \end{bmatrix}.$$

It is obvious that $A(t)$ is continuous, and from [8], the eigenvalues of $A(t)$ are

$$0.5(\alpha\sin(t)-2) \pm \sqrt{\alpha^2\cos^2(t)+4\alpha(\sin(t)-\cos(t))-4},$$

which implies that $A(t)$ is bounded for any given $\alpha > 0$, hence the satisfaction of assumption on $A(t)$.Therefore, the proposed integral function approach is applicable to the exponential stability analysis of this example.

The basis consist of $z_1 = (1,0)^T$ and $z_2 = (0,1)^T$ is first chosen**.** Then, the Algorithm 1 is employed to compute the integral function on z_1 and z_2 for different α and $\lambda > 0$. It can be found that for any given $\alpha < 3.16$, there exists $\lambda > 1$ satisfying $I_\lambda(z_{1,2})$ are finite, which indicates that the LTV system in Example 1 is exponential stable according to Theorem 1. This result is quite consistent with [10], and hence demonstrates the efficacy of the integral function approach.

Furthermore, for any given $\alpha < 3.16$ in this example, the exponential decay rate can be obtained. For instance $\alpha = 3$, $I_\lambda = \max\{I_\lambda(z_1), I_\lambda(z_2)\}$ for $\lambda = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 are computed by Algorithm 1. The obtained is plotted in Fig. 1, an estimation of $\lambda^* = 1.082$ can be obtained by the extrapolation method, thus the exponential decay rate is provided as $r^* = (\lambda^*)^{-1/2} = 0.9614$ according to Theorem 2. This result was never provided in any existing references, which illustrates the advantage of the proposed integral function approach.

The next example illustrates that the proposed integral function approach can also be useful in analysis of the unstable LTV systems. It is effective to detect the instability, also show its exponential divergence rate.

Example 2: The second example is an unstable system, which has been proved in [8,10,14]. The system matrix is given by

$$A(t) = \begin{bmatrix} -1+1.5\cos^2(t) & 1-1.5\sin(t)\cos(t) \\ -1-1.5\sin(t)\cos(t) & -1+1.5\sin^2(t) \end{bmatrix}.$$

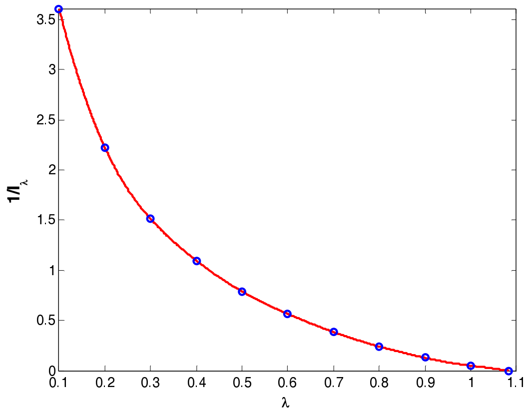

Fig. 1. Simulation results of Example 1.

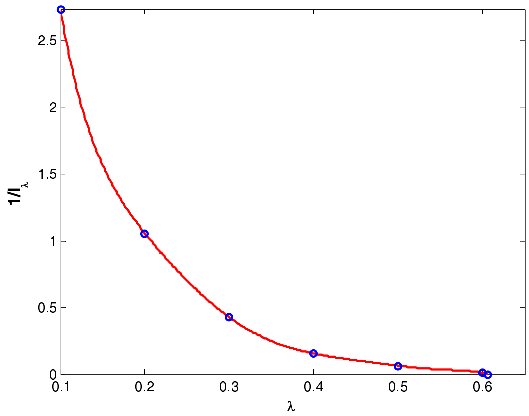

Fig. 2. Simulation results of Example 2.

It can be easily verified that $A(t)$ is continuous and bounded, i.e., the assumption on $A(t)$ is satisfied. Similar to Example 1, the standard basis is chosen as $z_1 = (1,0)^T$ and $z_2 = (0,1)^T$. Then, Algorithm 1 is used to compute the $I_\lambda = \max\{I_\lambda(z_1), I_\lambda(z_2)\}$ for λ = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35 and 0.4, respectively. Since I_λ at $\lambda = 0.4$ is infinite, hence the LTV system in this example is unstable from Theorem 1. This result is consistent with [8,10,14].

Furthermore, the computed results on I_λ are plotted in Fig. 2 such that an estimated of $\lambda^* = 0.3679$ can be obtained by the extrapolation method, thus the exponential divergence rate is provided as $r^* = (\lambda^*)^{-1/2} = 1.648$ according to Theorem 2.

5. CONCLUSION

By applying the Bellman-Gronwall lemma, this paper extended the recent proposed integral function approach to the exponential stability analysis of the linear time-varying (LTV) systems. It was found that the improved integral function can fully characterize the exponential stability of LTV systems, and the exponential decay rate of system trajectories can be exactly computed by the radius of convergence of the integral function. Moreover, the algorithm for the computation of the integral function is developed, and two classical examples illustrate the efficacy and advantage of the proposed approach.

Furthermore, we will consider the possibility to extend the integral function approach to the robust stability analysis of LTV systems with time-varying polytopic uncertainty, the key idea is to consider such kind of LTV systems as a special class of time-varying switching systems with infinite switching modes, then characterize its exponential stability using a new class of integral functions. Such a possibility is currently under study.

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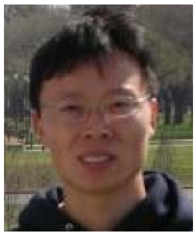

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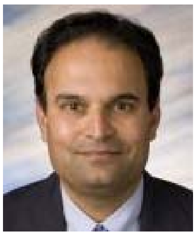

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